University of Science and Technology of
Oran "Mohamed Boudiaf"

FACULTY OF MECHANICAL
ENGINEERING

COURSE SUPPORT

Matter to teach

Numerical methods and programming techniques
“Properties of matrices and systems of linear equations”

PRESENTED BY

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Summary

I. Introduction

This document is not intended to be a course of all numerical methods, just a course on matrices. The idea is to give at the very beginning of this document the essentials on matrices. To treat the methods of resolution of the systems of linear algebraic equations thereafter. Certainly, in the literature many methods exist, there are direct methods and iterative methods. We will approach in this support only the direct methods which will allow us to solve the systems of linear algebraic equations.

The most commonly used direct methods are direct elimination, Cramer's rule, Gaussian elimination, Gauss-Jordan elimination, matrix inversion and matrix factorization. At the beginning, this course intended for the third year license aimed to inculcate for students the programming techniques, particular the programming language FORTRAN and MATLAB. Noting huge gaps in our students, in particular matrix operations and numerical methods, the content of this course has been improved and adapted for the training needs of this course. In order to provide the student with course materials, this handout was produced and mainly inspired by the book “Numerical Methods for Engineers and Scientists” by author Joe D. Hoffman. The latter is devoted solely to the part of matrix operations and numerical methods for solving systems of linear equations.

II. Properties of matrices and determinants

Systems of linear algebraic equations can be well expressed in terms of matrix notation. Methods for solving systems of linear algebraic equations can be developed in a very compact way using matrix algebra. Therefore, the elementary properties of matrices and determinants are presented in this section.

III. Matrix Definition

A matrix is a rectangular array of elements (numbers or symbols), which are arranged in ordered rows and columns. Each element of the matrix is distinct with separation. The location of an element in the matrix is important. The elements of a matrix are generally identified by an indexed double lowercase letter, for example, a_{ij}, such that the first index i identifies the row of the matrix and the second index j identifies the column of the matrix. The size of the matrix is indicated by the product of the number of rows by the number of columns. A matrix with n rows and m columns is said to be n by m or nxm matrix. Arrays are usually represented by either a bold uppercase letter, e.g., A, or the element in square brackets, e.g., $[a_{ij}]$, or the full range of elements, as shown in the equation:

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad (i = 1,2, \ldots \ldots, n\,; j = 1,2, \ldots \ldots, m) \qquad (1)$$

This notation is used throughout this handout for a simpler appearance. When the general element $a_{i,j}$ is considered, the indices i and j are separated by a comma. When a specific element is used, for example, a_{14}, the subscript 1 and 4, which denote the element in row 1 and column 4, will not be separated by a comma, except for i or j greater than 9. For example, an a_{68} denotes the item in row 6 and column 8, while $a_{14,5}$ denotes the item in row 14 and column 5.

1. Matrix vector

Vectors are a special type of matrix that only has one column or one row. Vectors are represented by either a bold lowercase letter, e.g., x or y, the item in square brackets $[x_i]$ or $[y_i]$ denotes the full column or full row of items. The column vector is $a_{n\ n} \times 1$ matrix:

$$X = [x_i] = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad (i = 1,2, \dots \dots, n) \tag{2}$$

The row vector is a matrix of 1× n:

$$Y = [y_i] = [y_1 \quad y_2 \quad \cdots \quad y_n] \quad (j = 1,2, \dots \dots, n) \tag{3}$$

Unit vectors, i, are special vectors that have a unity magnitude:

$$\|i\| = (i_1^2 + i_2^2 + \cdots + i_n^2)^{1/2} = 1 \tag{4}$$

Where the notation ‖i‖ denotes the length of vector i, orthogonal system for unit vector, in which all elements are zero, except one, and which are used to define coordinate systems.

2. **Square matrix**

There are several types of special matrices. A square matrix S is a matrix that has the same number of rows and columns (m = n):

$$S = \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ \mathrm{M} & \mathrm{O} & & \mathrm{M} \\ \mathrm{M} & & \mathrm{O} & \mathrm{M} \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \tag{5}$$

S is an n x n square matrix. Our interest will be devoted entirely to square matrices. The descending left-to-right line of elements from a_{11} to a_{nn} is called the diagonal of the matrix.

3. **Diagonal matrix**

A diagonal matrix is a square matrix whose coefficients outside the main diagonal are zero. The diagonal coefficients may or may not be zero. Any diagonal matrix is also a symmetric matrix. For example D is a diagonal matrix of 4 x 4.

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{bmatrix} \quad (6)$$

D is a diagonal matrix if and only if it satisfies:

$$\forall i, j \in \{1, \dots, n\}, i \neq j \;\Rightarrow\; d_{i,j} = 0$$

As a diagonal matrix is entirely determined by the list of its diagonal elements, the following more concise notation is often adopted:

$$diag(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \quad (7)$$

Diagonal matrices appear in almost all areas of linear algebra. The multiplication of diagonal matrices is very simple; also, if an interesting matrix can somehow be replaced by a diagonal matrix, then calculations involving it will be faster and the matrix easier to store in memory. A process for making certain matrices diagonal is diagonalization.

A diagonal matrix of order n naturally has eigencolumns which are coordinates of n orthonormal vectors and its diagonal coefficients are exactly the associated eigenvalues.

See also the singular value decomposition, according to which any matrix is unitarily equivalent to a zero-bounded positive diagonal matrix.

In other words, for all diagonal matrices $D = diag((d_i)_{1 \leq i \leq n})$ and

$E = diag((e_i)_{1 \leq i \leq n})$ we have :

- For everything $(\lambda, \mu) \in \Re^2,\ \lambda D + \mu E = F$ with $F = diag((\lambda d_i + \mu e_i)_{1 \leq i \leq n})$
- $DE = ED = G$ with $G = diag((d_i e_i)_{1 \leq i \leq n})$

A consequence of this is that raising a diagonal matrix D to a certain power amounts to raising the coefficients of the diagonal of D to that power:

$$D^k = diag(d_{i,j})^k = diag(d_{i,j}^k)$$

4. Scalar matrix

These are the diagonal matrices whose all diagonal coefficients are equal. For example:

$$\begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix} \tag{8}$$

The determinant of a diagonal matrix is equal to the product of its diagonal elements:

$$\det(diag(a_1, a_2, ..., a_n)) = \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{vmatrix} = \prod_{k=1}^{n} a_k \tag{9}$$

A diagonal matrix is invertible if and only if its determinant is nonzero, that is, if and only if all of its diagonal elements are nonzero. In this case, the inverse of a diagonal matrix is a diagonal matrix where the diagonal coefficients are the inverses of the diagonal coefficients of the starting matrix.

Indeed, if:

$$D = diag(a_1, a_2, ..., a_n) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \tag{10}$$

So

$$D^{-1} = diag(\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}) = \begin{bmatrix} 1/a_1 & 0 & \cdots & 0 \\ 0 & 1/a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/a_n \end{bmatrix} \tag{11}$$

Because

$$D.D^{-1} = D^{-1}.D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I_n \qquad (12)$$

Let the identity matrix

5. Identity matrix

The identity matrix or unit matrix is a square matrix with 1s on the diagonal and 0s everywhere else. It can be written diag(1, 1, …, 1).

Since matrices can be multiplied only if their types are compatible, there are unit matrices of any order. I_n is the unit matrix of order n and is therefore defined as a diagonal matrix with 1 on each entry of its main diagonal.

So:

$$I_1 = [1],\ \ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ldots,\ \ I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad (13)$$

Concerning the product of the matrices, the unit matrices verify that for all p, n non-zero natural integers and for any matrix A with n rows and p columns,

$$I_nA = AI_p = A,$$

This shows that the product by a unit matrix has no effect on a given matrix. This can be demonstrated by direct computation or by noticing that the identity map (which it represents in any basis) has no effect by composition with a given linear map.

In particular, I_n is the neutral element for the product of square matrices of order n.

It is also possible to denote the coefficients of the unit matrix of order n with the Kronecker symbol; the coefficient of the i-th row and j-th column is written:

$$\delta_{ij} = \begin{cases} 1 & si\ i = j \\ 0 & si\ i \neq j \end{cases}$$

And therefore the unit matrix I is equal to:

$$I = (\delta_{ij})$$

If the order is not specified, or it is trivially determined by the context, we can simply write it down *I*.

6. The null matrix

It is the not necessarily square matrix of which all the coefficients are zero. It is denoted $0_{n;\,p}$ or $0_{n\ p}$ if it has n rows and p columns. For example:

$$0_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

7. Triangular matrix

Triangular matrices are square matrices in which a triangular part of the values, bounded by the main diagonal, is zero.

By definition, an upper triangular matrix with real coefficients is a square matrix whose values under the main diagonal are zero:

$$U = (a_{i\,j}) = \begin{bmatrix} a_{1,1} & a_{1,2} & \mathrm{L} & \mathrm{L} & a_{1,n} \\ 0 & a_{2,2} & & & a_{2,n} \\ \mathrm{M} & \mathrm{O} & \mathrm{O} & & \mathrm{M} \\ \mathrm{M} & & \mathrm{O} & \mathrm{O} & \mathrm{M} \\ 0 & \mathrm{L} & \mathrm{L} & 0 & a_{n,n} \end{bmatrix} \tag{15}$$

A is upper triangular "U" if and only if:

$$\forall i > j, \;\; a_{i,j} = 0$$

By definition, a lower triangular matrix with real coefficients is a square matrix whose values above the main diagonal are zero:

$$L=(a_{i,j})=\begin{bmatrix} a_{1,1} & 0 & \cdots & \cdots & 0 \\ a_{2,1} & a_{2,2} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{bmatrix} \tag{16}$$

A is lower triangular "L" if and only if:

$\forall i < j,\ \ a_{i,j} = 0$

8. Triangular matrices properties

- A triangular matrix that is both lower and upper is a diagonal matrix.
- The sum of two lower (respectively upper) triangular matrices and their opposites are lower (respectively upper) triangular matrices.
- If we left or right multiply a lower (respectively upper) triangular matrix by a scalar, the result is again a lower (respectively upper) triangular matrix.
- The product of two lower (respectively upper) triangular matrices is a lower (respectively upper) triangular matrix.
- The identity matrix is a diagonal matrix and therefore both an upper triangular and a lower triangular matrix.
- The transpose of an upper triangular matrix is a lower triangular matrix, and vice versa.
- If A = $(a_{i,j})_{i,j}$ and B = $(b_{i,j})_{i,j}$ are upper triangular matrices with n rows and n columns with real coefficients, the i-th diagonal coefficient of AB is $a_{i,i}\, b_{i,i}$. In other words, the diagonal of the product AB is the product of component by component of the diagonals of A and B.
- Let A be an upper (respectively lower) triangular matrix of size n, if all the diagonal coefficients of A are invertible, the matrix A is invertible. In this case, its inverse is also an upper (respectively lower) triangular matrix. It follows that the diagonal coefficients of the inverse of A are then the inverses of the diagonal coefficients of A.

9. Transposed matrix

The transpose matrix (we also say the transpose) of a matrix A∈M_(m,n) (K) is the matrix denoted A^T, obtained by exchanging the rows and columns of *A*.

If $B = {}^{t}A$ then $\forall (i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}, b_{i,j} = a_{j,i}$.

Example:

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ then ${}^{t}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

- The "transposition" map is linear:

$${}^{t}(A+B) = {}^{t}A + {}^{t}B,$$

$${}^{t}(\alpha A) = \alpha\,{}^{t}A$$

- The transpose of ${}^{t}A$ is A.
- The transpose of the product of two matrices is equal to the product of the transposes of these two matrices, but in reverse order:

$${}^{t}(AB) = {}^{t}B\,{}^{t}A$$

- If a square matrix A is invertible, then its transpose is also, and the transpose of the inverse of A is equal to the inverse of its transpose:

$${}^{t}(A^{-1}) = ({}^{t}A)^{-1}$$

- If A denotes a square matrix of size n and B its transpose, then A and B have the same main diagonal (and therefore the same trace):

$$b_{ii} = a_{ii}$$

- In particular, any diagonal matrix is symmetric, that is to say equal to its transpose.
- More generally, two square matrices transposed from each other have the same characteristic polynomial and therefore the same eigenvalues, the same trace but also the same determinant.

IV. Matrix operations

1. Sum and difference

The sum or difference between two matrixes is a very simple operation. It is simply noted + or – and we have the following definition:

Let A and B be two matrixes having the same size, then if we have:

$$A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

$$B = (b_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

So

$$A + B = (a_{ij} + b_{ij})$$

You see that the definition is precise. The addition of two matrixes is only possible provided that the two matrixes have the same size i.e. the same number of rows and the same number of columns. Otherwise, the sum is not possible.

Example:

Either $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix} \quad et \quad B = \begin{bmatrix} O & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix}$

These two matrixes both have the same size. The addition is therefore possible and we have:

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} O & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 \\ 3+(-1) & 1+(-1) \\ 1+1 & 0+2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}$$

You can see that the sum of two matrixes is commutative A + B = B + A, it is also associative $(A + B) + C = A + (B + C)$

However:

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} O & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 0 & -2 \end{bmatrix}$$

And

$$B - A = \begin{bmatrix} O & 1 \\ -1 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -4 & -2 \\ 0 & 2 \end{bmatrix}$$

So $A - B = -(B - A)$

2. Product of a scalar by a matrix

We can multiply a matrix by a scalar α, that is to say an element of the set of real.
Consider a matrix A such that A = (a_{ij}) and a scalar α;

$$\alpha A = (\alpha a_{i\,j}) = \begin{bmatrix} \alpha a_{11} & \mathrm{L} & & \mathrm{L} & \alpha a_{1p} \\ \mathrm{M} & \mathrm{O} & & & \mathrm{M} \\ & & \alpha a_{i\,j} & & \\ \mathrm{M} & & & \mathrm{O} & \mathrm{M} \\ \alpha a_{n\,1} & \mathrm{L} & & \mathrm{L} & \alpha a_{n\,p} \end{bmatrix} \qquad (17)$$

Either

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

And α = 7 then

$$7A = \begin{bmatrix} 7 \times 1 & 7 \times 2 \\ 7 \times 3 & 7 \times 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$$

Several remarks on this operation:

- It is possible not to write · the multiplication; so we write αA rather than $\alpha \cdot A$
- The scalar is always written on the left, so we write 7A but not $A7$.
- In the same way we write 1/7 A but not $\frac{A}{7}$
- The product of a scalar by a matrix is an external law.
- $\alpha(A + B) = \alpha A + \alpha B$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$

3. Product of two matrix

a) Product of a row matrix by a column matrix

Or $A = (a_1, \dots, a_p)$ a line matrix and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_P \end{pmatrix}$ a column matrix (note that the row matrix and the column matrix have the same number of elements). The product of A by B, noted AB is matrix 1x1 which is a scalar C:

$$C = a_1 b_1 + a_2 b_2 + \cdots + a_p b_p$$

Example:

The product of $A = (2 \quad 0 \quad 1)$ by $B = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is

$$C = A \times B = (2 \quad 0 \quad 1) \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2 \times 2 + 0 \times (-1) + 1 \times 0 = 4$$

b) Product of a matrix by a column matrix

Let A be a matrix of n rows and p columns and B a column matrix of p rows; note for the product to be possible, the matrix A must have as many columns as B has rows.

The product of A by B, denoted AB, is the n × 1 column matrix whose row number i is the scalar resulting from the product of row number i of A with column B for each row number i between 1 and n :

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ip} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{np} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1p} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \\ \vdots \\ \begin{pmatrix} a_{i1} & \cdots & a_{ip} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \\ \vdots \\ \begin{pmatrix} a_{n1} & \cdots & a_{np} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + \ldots + a_{1p}b_p \\ \vdots \\ a_{i1}b_1 + \ldots + a_{ip}b_p \\ \vdots \\ a_{n1}b_1 + \ldots + a_{np}b_p \end{bmatrix} \qquad (18)$$

Example:

Let the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ to multiply by the column matrix $B = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$:

$$A \times B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 1 + 0 \times 3 + 1 \times 0 \\ 3 \times 1 - 1 \times 3 + 2 \times 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

c) Product of a matrix by a matrix

To multiply two matrices A and B, so that this operation becomes feasible: it should be noted that A has as many columns as B has rows.

The product of A[n,p] by B[p,q] is the matrix C[n,q] whose column number j is the product of A by column number j of B for each column number j between 1 and q: generally speaking, if the two matrices $A = (a_{i\,j})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$ et $B = (b_{i\,j})_{\substack{1 \le i \le p \\ 1 \le j \le q}}$

So for everything $i \in \{1, \dots, n\}$ and all $j \in \{1, \dots, q\}$; $C = A \times B : c_{i\,j} = \sum_{k=1}^{p} a_{i\,k} b_{k\,j}$

Example:

We want to produce the product of $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ by $B = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$.

Let's start by noticing that A has three columns and B has three rows: the product can be calculated. Furthermore, A has 2 rows and B 4 columns, the product matrix C will therefore be of size [2, 4].

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} (2 \times 2 + 0 \times (-1) + 1 \times 0) & (2 \times 0 + 0 \times 1 + 1 \times 1) & (2 \times 1 + 0 \times 3 + 1 \times 0) & (2 \times 0 + 0 \times 1 + 1 \times 2) \\ (3 \times 2 + (-1) \times (-1) + 2 \times 0) & (3 \times 0 + (-1) \times 1 + 2 \times 1) & (3 \times 1 + (-1) \times 3 + 2 \times 0) & (3 \times 0 + (-1) \times 1 + 2 \times 2) \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 1 & 2 & 2 \\ 7 & 1 & 0 & 3 \end{bmatrix}$$

The product of matrix A and B is only defined if the number of columns of A is equal to the number of rows of B.

The following properties can be demonstrated computationally using the expression

$$C = A \times B : c_{i\,j} = \sum_{k=1}^{p} a_{i\,k} b_{k\,j}$$

Let A, B and C be such that

- The number of columns in A is equal to the number of rows in B (so we can calculate AB);
- The number of columns in B is equal to the number of rows in C (so we can calculate BC).

Then the operation of the matrix product is associative (AB)C = A(BC).

The matrix product is not commutative. This is obvious when we can calculate AB but not BA (which happens if the number of columns of A is equal to the number of rows of B but the number of columns of B differs from the number of rows of A) but we can also have AB≠BA when A and B are two square matrices of the same order. So, for

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad et \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We have $AB = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$ but $BA = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$

If A is a matrix of n rows and p columns, we have

$AI_p = A$ et $I_nA = A$.

If A, B and C are three matrixes such that A and B have the same number of rows and the same number of columns and the number of rows of C is equal to the number of columns of A (and therefore of B), then

(A+ B)C = AC+ BC.

If A, B and C are three matrixes such that B and C have the same number of rows and the same number of columns and the number of columns of A is equal to the number of rows of B (and therefore of C), then

A(B+ C) = AB+ AC.

If A and B are two matrixes such that the number of columns of A is equal to the number of rows of B and if λ is a real coefficient, then

$$\lambda (AB) = (\lambda A)B = A(\lambda B).$$

The product of two matrixes can be zero while neither matrix is zero. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \quad et \;\; B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So AB=0 but A≠0 et B≠0.

4. Power

We define the power k of the square matrix A of size n for the integer $k \geq 0$ as follows:

$$A^k = \begin{cases} I_n & \text{si k=0} \\ A^{k-1}A = \underbrace{A \ldots A}_{k \text{ fois}} & \text{si } k \geq 1 \end{cases}$$

Example:

Supposedly $A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{bmatrix}$, then

$$A^3 = A^2 A = \begin{bmatrix} 110 & 134 & 164 \\ 312 & 389 & 477 \\ 558 & 697 & 861 \end{bmatrix} A = \begin{bmatrix} 3946 & 4920 & 6064 \\ 11456 & 14278 & 17588 \\ 20632 & 25700 & 31654 \end{bmatrix}$$

Let A be a square matrix, let k and l be two integers

$$A^k A^l = A^{k+l}$$

$$(A^k)^l = A^{kl}$$

$$(\alpha A)^k = \alpha^k A^k$$

5. Inverse of a square matrix

If x is a non-zero real, it admits an inverse: it is a real $y = 1/x$ such that $xy = 1$ and by commutativity, $yx = 1$. Since multiplication is not commutative in Mn(K), it Precautions must first be taken.

Let A be a square matrix of order n. A square matrix B of order n is called right inverse of A if $AB = I_n$ and left inverse of A if $BA = I_n$.

If a matrix A admits a right inverse B and a left inverse C then $B = C$ and we can therefore say that B is an inverse of A without ambiguity. Let's show it by calculating CAB in two ways thanks to the associativity of the matrix product: $(CA)B = C(AB)$ therefore $I_nB = CI_n$ then $B = C$.

Furthermore, if a matrix A admits an inverse on the right and on the left, this inverse is unique. Suppose that B and C are two inverses to the right and to the left: $AB = BA = I_n$ and $AC = CA = I_n$. Then $B = C$ because

$$C = CI_n = C(AB) = (CA)B = I_nB = B.$$

A square matrix is said to be invertible if it admits an inverse on the right and on the left. Its inverse is then unique. We notice A^{-1} the inverse of the invertible matrix A.

If A a square matrix of order n, we have $AA^{-1} = A^{-1}A = I_n$.

Let A an invertible matrix and λ any coefficient:

- The matrix A^{-1} is invertible from inverse A.
- The matrix λA is invertible from inverse $\frac{1}{\lambda}A^{-1}$.

Let A and B be two invertible square matrices of the same size. Then the product AB is invertible and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.

Be careful of changing the order of multiplication when taking the inverse of a product.

If a square matrix admits an inverse on the left, and admits an inverse on the right, it is therefore invertible. Given a square matrix A, if we find a matrix of the same size B such that $AB = I$ then $BA = I$ and $B = A^{-1}$.

We can then speak of negative powers of an invertible matrix. If A is invertible then A^k is invertible for any integer $k \geq 0$ with inverse $(A^{-1})^k$.

Then we pose $A^{-k} = (A^{-1})^k = (A^k)^{-1}$.

Let A be a square matrix of size n. We construct a matrix with n rows and $2n$ columns $(A \mid I)$ by writing the identity matrix of order n to the right of A. By applying elementary operations, we transform the matrix A into the identity matrix. We apply the same operations to I. We transform A into I, The same elementary operations transform I into A^{-1}.

Example:

We seek to invert the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$. We are therefore working on

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$L_2 \leftarrow \frac{1}{2} L_2 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$L_2 \leftarrow L_2 - L_1 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$L_3 \leftarrow L_3 + L_1 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{cases} L_2 \leftarrow -\frac{2}{3} L_2 \\ L_3 \leftarrow \frac{1}{3} L_3 \end{cases} \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$L_3 \leftarrow L_3 - L_2 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$L_3 \leftarrow -3 L_3 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$L_1 \leftarrow L_1 - L_2 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$L_1 \leftarrow L_1 - \frac{2}{3}L_3 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 1 & 2/3 \\ 0 & 1 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$L_2 \leftarrow L_2 - \frac{1}{3}L_3 \quad : \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 1 & 2/3 \\ 0 & 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] = (I \vdots A^{-1})$$

So we have

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/3 & 1 & 2/3 \\ 1/3 & 0 & 1/3 \\ 1 & -1 & -1 \end{bmatrix}$$

Example:

We seek to invert the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{bmatrix}$. We are therefore working on

$$[A \quad \vdots \quad I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 4 & 7 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$L_2 \leftarrow L_2 - 2L_1 : \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 4 & 7 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$L_3 \leftarrow L_3 - 4L_1 : \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 6 & -4 & 0 & 1 \end{array}\right]$$

$$L_3 \leftarrow L_3 - 3L_2 : \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 1 \end{array}\right].$$

Before arriving at the identity matrix, the last row of the matrix is zero, so matrix A is not invertible. Each square matrix is associated with a number to allow us to determine if the matrix is invertible, this number is called determinant.

6. Determinant

We associate with each square matrix a number allowing us to determine whether it is invertible: the determinant. The determinant of a matrix is only defined if the matrix is square. Let A be a square matrix of size n. The determinant of A, denoted $\det(A)$, is a real number defined by "descent" as follows:

a) if $n = 1$ then $A = (a_{11})$ and $\det(A) = a_{11}$;

b) if $n \geq 2$, then $A = (a_{ij})_{1\leq i;\, j \leq n}$ and

$$\det(A) = a_{11}\Delta_{11} - a_{21}\Delta_{21} + a_{31}\Delta_{31} - \ \ldots + (-1)^{n-1}a_{n1}\Delta_{n1}$$

Where Δ_{i1} is the determinant of the matrix of size $n-1$ obtained by removing row numbers i and the first column from A.

We consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we calculate the determinant of A:

$$\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\det((d)) - c\det((b)) = ad - bc$$

The matrix A is invertible if and only if the quantity $ad - bc$ is non-zero.

Example:

We calculate

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \times \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

$$= (5 \times 9 - 6 \times 8) - 4 \times (2 \times 9 - 3 \times 8) + 7 \times (2 \times 6 - 3 \times 5) = 0$$

We can calculate the determinant in another way, especially for the case of a matrix of size $n \geq 3$, we add additional columns to our matrix, more exactly $n-1$ columns. The added columns are just the columns of our matrix, starting with the first column up to column $n-1$, placing them after the last column, so that the new matrix becomes rectangular with n rows and $2n-1$ columns.

For the case of a square matrix A of size n=3, by adding the 2 columns we find a matrix of size 3×5 so as to have 3 diagonals which are connected by a blue line and 3 anti-diagonals which are connected by dashes in red:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example:

Let's calculate the determinant of the following square matrix using the diagonal method:

$$A = \begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix}$$

By adding the first two columns, we increase our matrix as follows:

$$A = \begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} \begin{matrix} 80 & -20 \\ -20 & 40 \\ -20 & -20 \end{matrix}$$

$$\begin{aligned} \det(A) &= (80)(40)(130) + (-20)(-20)(-20) + (-20)(-20)(-20) \\ &\quad - (-20)(40)(-20) - (-20)(-20)(80) - (130)(-20)(-20) \\ &= 416000 - 8000 - 8000 - 16000 - 32000 - 52000 \\ &= 300000 \end{aligned}$$

The determinant of a triangular coefficient matrix in R is the product of its diagonal coefficients:

$$A = (a_{ij}) = \begin{bmatrix} a_{1,1} & a_{1,2} & \mathrm{L} & \mathrm{L} & a_{1,n} \\ 0 & a_{2,2} & & & a_{2,n} \\ \mathrm{M} & \mathrm{O} & \mathrm{O} & & \mathrm{M} \\ \mathrm{M} & & \mathrm{O} & \mathrm{O} & \mathrm{M} \\ 0 & \mathrm{L} & \mathrm{L} & 0 & a_{n,n} \end{bmatrix} \text{ ou } A = (a_{i,j}) = \begin{bmatrix} a_{1,1} & 0 & \mathrm{L} & \mathrm{L} & 0 \\ a_{2,1} & a_{2,2} & \mathrm{O} & & \mathrm{M} \\ \mathrm{M} & & \mathrm{O} & \mathrm{O} & \mathrm{M} \\ \mathrm{M} & & & \mathrm{O} & 0 \\ a_{n,1} & a_{n,2} & \mathrm{L} & \mathrm{L} & a_{n,n} \end{bmatrix}$$

$$\det((a_{i,j})_{i,j\in\{1,n\}}) = \prod_{i=1}^{n} a_{i,i}$$

Example:

We calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$$\det A = 1 \times 4 \times 6 = 24$$

The identity matrix I_n its determinant equal to 1, if any matrix has a row (respectively a column) made up of zero elements then its determinant is zero.

The diagonal method of evaluating determinants only applies to matrices of size 2x2 and 3x3. It is no longer valid for matrices of size 4x4 or larger. In general, the expansion by a factor of *n*x*n* is the sum of all possible products formed by the choice of one and only one element of each row and each column of the matrix, with a sign more or less determined by the number of permutations of the row and column elements.

A formal procedure for evaluating determinants is called minor extension, or the cofactor method. In this procedure, there is an «n!» product to add them, where each product has n elements. So expanding by a factor of 10 x 10 requires adding "10!" products (10! = 3628800), where each product involves 9 multiplications (the product of 10 elements).

That's a total of 32,659,000 multiplications and 3627999 additions, not counting the work needed to keep track of the signs.

Consequently, the evaluation of determinants by the cofactor method is not possible, except for very small matrices. Although the cofactor method is not recommended for any type of matrix larger than 4x4, it is helpful to understand the concepts.

The minor $M_{i\,j}$ is the determinant of the sub matrix (n - 1).(n - 1) of the nxn matrix *A* obtained by deleting row *i* and column *j*, The cofactor $A_{i\,j}$ associated with the minor $M_{i\,j}$ is defined as:

$A_{i\,j} = (-1)^{i+j} M_{i\,j}$

Using cofactors, the determinant of matrix *A* is the sum of the products of the elements in a row or column, multiplied by their corresponding cofactors. Thus, by expanding along any fixed line *i*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{i\,j} A_{i\,j} = \sum_{j=1}^{n} (-1)^{i+j} a_{i\,j} M_{i\,j}$$

Alternatively, developing calculations down the fixed columns *j*

$$\det(A) = |A| = \sum_{i=1}^{n} a_{i\,j} A_{i\,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i\,j} M_{i\,j}$$

Each cofactor expansion reduces the order of the determinant by one, so there are *n* matrixes of order *n*-1 to evaluate their determinants. By repeated application, the cofactors are finally reduced to 3 × 3 determinants which can be evaluated by the diagonal method.

The volume of work can be reduced by choosing the expansion row or column with as many zeros as possible.

If the value of the determinant of the matrix is equal to zero, the matrix is said to be singular. There non-singular or invertible matrix has the determinant that has a non-zero value. If a row or column of a matrix consists of zero elements, this matrix is singular.

The determinant of an upper triangular or lower triangular matrix is the product of the elements of the main diagonal. It is possible to transform any invertible matrix into a triangular matrix, such that the value of the determinant remains unchanged, no matter how many operations are necessary for this transformation. This method involves transforming the matrix into an upper triangular matrix or a lower triangular matrix. The value of the determinant of the triangular matrix can then be evaluated quite easily by the product of the elements of the main diagonal.

V. Systems of linear algebraic equations

Systems of equations arise in all branches of engineering and science. This part is devoted to the solution of systems of linear algebraic equations of the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ \dots + a_{2n}x_n = b_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ \dots + a_{nn}x_n = b_n \end{cases} \qquad (19)$$

Where x_j (j = 1, 2 ..... n) denotes the unknown variables to be determined, $a_{i,\,j}$ (i, j = 1, 2 ..... n) denotes the constant coefficients of the unknown variables, and b_i (i = 1, 2 ..... n) the column matrix, denotes the non-homogeneous terms, is called the second member of the system. For the coefficients $a_{i,\,j}$, the first index, i, denotes equation i and the second index, j, denotes variable x_j. The number of equations can vary from two to hundreds, thousands, or even millions.

In the general case, the number of variables does not have to be the same as the number of equations. However, in most practical problems, they are the same, namely the case considered in this handout. When the number of variables is the same as the number of equations, a single solution may exist, as illustrated by the example of a system of two algebraic linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{20}$$

If the second member of the system (20) B = 0, the system is said to be a system of homogeneous linear equations. Systems of homogeneous linear equations always admit at least one solution: the null column matrix.

There are two fundamentally different approaches to solving algebraic linear equations (19):

- Direct elimination methods
- Iterative methods

Direct elimination methods are systematic procedures based on algebraic elimination, which obtain the solution in a finite number of operations. Direct elimination methods include: Gauss elimination, Gauss-Jordan elimination, inverse matrix method and LU factorization.

On the other hand, iterative methods obtain the solution asymptotically by an iterative procedure. A test solution is assumed, the test solution is replaced in the system of equations to determine the mismatch or error, in the test solution, and an improved solution is obtained from the inadequate data. Iterative methods are the Jacobi iterative method, the Gauss-Seidel iterative method and successive-on-relaxation (SOR).

Although there are no strict application rules, direct elimination methods are generally used when one or more of the following conditions are met:

a) The number of equations is small (100 or less),

b) Most coefficients in the equations are nonzero,

c) When the system of equations is not diagonally dominant or

d) The system of equations is poorly conditioned.

Iterative methods are used when the number of equations is large and most coefficients are zero (i.e., a hollow matrix). Iterative methods generally diverge unless the system of equations is diagonally dominant.

VI. Direct elimination Methods

There are a number of methods for the direct resolution of systems of linear equations. One of the best known methods is the Cramer rule, which requires the determinant calculation. Methods based on the elimination concept are also recommended.

Elimination rules and methods are presented in this section. After the presentation of the Cramer rule, Gauss elimination, Gauss-Jordan elimination and matrix inversion; these concepts are extended to the LU factorization and three-dimensional systems of equations in the following sections.

1. The Rule of Cramer

Although it is not an elimination method; the Cramer rule is a direct method for solving systems of algebraic linear equations. Consider the system of equations, Ax = b, which represents *n* equations A square matrix of size *n*. The Cramer rule states that the solution for x_j (j = 1 ..... n) is given by

$$x_j = \frac{\det(A^j)}{\det(A)} \qquad (j = 1, \dots, n) \qquad (21)$$

Where A^j is the matrix of size *n* obtained by replacing column j of matrix *A* by column b. For example, consider the system of two equations (20), applying the Cramer rule:

$$x_1 = \frac{\det\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

$$x_2 = \frac{\det\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

The determinants can be evaluated by the diagonal method described above. For systems containing more than three equations i.e. matrix of size $n \geq 4$, the evaluation of the determinant requires another method of calculation called cofactors process, also described earlier.

The number of operations required by this method is phenomenal (For a relatively small system containing 10 equations, the number of operations is 360,000,000, which is a huge number of calculations). This is obviously ridiculous to use the diagonal method for systems of equations of $n \geq 4$.

The preferred method for the evaluation of determinants, large matrices, is the elimination method in order to transform the matrix into upper or lower triangular matrix, which significantly reduces the number of operations.

The number of operations required by the elimination method or any matrix transformation method into a triangular matrix is about 1090 for n = 10. It is obvious that the elimination method is a less expensive method, and therefore the preferred.

Let us now illustrate the steps of the Cramer rule by solving the three-equation system:

$$\begin{cases} 80x_1 - 20x_2 - 20x_3 = 20 \\ -20x_1 + 40x_2 - 20x_3 = 20 \\ -20x_1 - 20x_2 + 130x_3 = 20 \end{cases} \tag{22}$$

First, calculating the determinant of A:

$$\det(A) = \det\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} = 300\,000$$

Subsequently, calculating the determinants of matrices A^1, A^2 and A^3, these matrixes are obtained by changing the columns of A by the values of b:

$$\det(A^1) = \det\begin{bmatrix} 20 & -20 & -20 \\ 20 & 40 & -20 \\ 20 & -20 & 130 \end{bmatrix} = 180\,000$$

$$\det(A^2) = \det\begin{bmatrix} 80 & 20 & -20 \\ -20 & 20 & -20 \\ -20 & 20 & 130 \end{bmatrix} = 300\,000$$

$$\det(A^3) = \det\begin{bmatrix} 80 & -20 & 20 \\ -20 & 40 & 20 \\ -20 & -20 & 20 \end{bmatrix} = 120\,000$$

This makes it possible to find the values of the three unknowns very easily by the Cramer rule

$$x_1 = \frac{\det(A^1)}{\det(A)} = \frac{180\ 000}{300\ 000} = 0{,}60$$

$$x_2 = \frac{\det(A^2)}{\det(A)} = \frac{300\ 000}{300\ 000} = 1$$

$$x_3 = \frac{\det(A^3)}{\det(A)} = \frac{120\ 000}{300\ 000} = 0{,}40$$

2. Elimination Method

The elimination methods aim to solve a system of linear equations by solving an equation, which is the first equation, for one of the unknowns, which is x_1, in terms of the remaining unknowns, x_2 to xn, and then replacing the expression of x_1 in the (n-1) equations to determine n - 1 equations involving x_2 to x_n. This elimination procedure is performed n - 1 times until the last step results in an equation involving only x_n. This process is called elimination.

The value of x_n can be calculated from the final equation of the elimination procedure. Then x_{n-1} can be calculated from the modified equation n - 1, which contains only x_n, and x_{n-1}. Then x_{n-2} can be calculated from the modified equation n-2, which contains only x_n, x_{n-1}, x_{n-2}. This procedure is executed n - 1 times for the calculation of x_{n-1}, at x_1. This process is called the return substitution.

3. Line operations

The elimination process uses the repetitive operations, which are:

1. 1. Any line can be multiplied by a constant.
2. 2. The order of the lines can be changed (swiveling).
3. 3. Any line may be replaced by a weighted linear combination of that line with one of the lines.

These operations on the lines, which change the values of the elements of the matrix A and b, do not change the solution x for the system of equations.

The first line operation is used at the line scale, if necessary. The second Line operation is used to avoid divisions by zero and reduce rounding errors. The third line operation is used to implement the elimination process.

4. Elimination example

Let us illustrate the method of elimination by solving the equation system (22):

Solve the first equation for x_1 as follows:

$$x_1 = (20 - (-20)x_2 - (-20)x_3)/80$$

By replacing the value of x_1 in the second equation of the system, we find

$$-20[(20 - (-20)x_2 - (-20)x_3)/80] + 40x_2 - 20x_3 = 20$$

This can be simplified to give

$$35x_2 - 25x_3 = 25$$

Similarly, by replacing the value of x_1 in the third equation of the system, we find

$$-20[(20 - (-20)x_2 - (-20)x_3)/80] - 20x_2 + 130x_3 = 20$$

This can be simplified to give

$$-25x_2 + 125x_3 = 25$$

Repeating the same operations for the second unknown x_2:

$$x_2 = (25 - (-25)x_3)/35$$

By replacing the value of x_2 in the last equation, we find

$$-25[(25 - (-25)x_3)/35] + 125x_3 = 25$$

At the end, we discover the final equation that concludes the elimination process.

$$\frac{750}{7}x_3 = \frac{300}{7}$$

This equation allows us to easily find the value of x_3.

By replacing by returning the value of x_3, the value of x_2 is determined and then that of x_1.

$x_3 = 300/750 = 0.40$

$x_2 = [25 – (–25)(0.40)]/35 = 1.00$

$x_1 = [20 – (–20)(1.00) – (–20)(0.40)]/80 = 0.60$

I am currently working on solving the equation system (22) in a standardized manner. The idea is to eliminate the coefficients of x_1 for equations two and three while keeping the coefficient of x_1 in the first equation, which is called the pivot. A multiplier is chosen to eliminate the coefficients below the pivot.

$$\begin{bmatrix} 80x_1 - 20x_2 & -20x_3 = 20 \\ -20x_1 + 40x_2 & -20x_3 = 20 \\ -20x_1 - 20x_2 & +130x_3 = 20 \end{bmatrix} \begin{matrix} \\ L_2 - (-20/80)\, L_1 \\ L_3 - (-20/80)\, L_1 \end{matrix}$$

After the first elimination, our system becomes to eliminate the new coefficient of x_2 from the third equation.

$$\begin{bmatrix} 80x_1 - 20x_2 & -20x_3 = 20 \\ 0x_1 + 35x_2 & -25x_3 = 25 \\ 0x_1 - 25x_2 & +125x_3 = 25 \end{bmatrix} \begin{matrix} \\ \\ L_3 - (-25/35)\, L_2 \end{matrix}$$

So the result after the second elimination is:

$$\begin{bmatrix} 80x_1 - 20x_2 & - & 20x_3 = & 20 \\ 0x_1 + 35x_2 & - & 25x_3 = & 25 \\ 0x_1 + \ 0x_2 & + & 750/7x_3 = & 300/7 \end{bmatrix}$$

This process continues until all coefficients below the main diagonal are eliminated. In our example with three equations, this process is now complete; the last system is the final result. This is the elimination process.

At this point, the last equation contains only one unknown, x_3, which can be solved. Using this result, the penultimate equation can be solved for x_2. Using the results of x_3 and x_2, the first equation can be solved for x_1.

$x_3 = 300/750 = 0.40$

$x_2 = [25 – (–25)(0.40)]/35 = 1.00$

$x_1 = [20 – (–20)(1.00) – (–20)(0.40)]/80 = 0.60$

This is the return substitution process. Thus, extending the elimination procedure for n equations is simple.

5. Simple elimination

The elimination procedure is illustrated in the above example involves the handling of coefficient of matrix A, and the non-homogeneous vector b. The components of vector X are fixed at their location in the set of equations. As long as the columns are not interchangeable, column j corresponds to x_j.

Therefore, the x_j notation does not need to be performed throughout the operations. Only the numerical elements of A and b should be considered. Thus, the elimination procedure can be simplified by increasing the matrix A by the vector of b and performing operations on the lines of the elements of the matrix A increased to complete the elimination process, then perform the return substitution process to determine the solution vector. This simplified elimination procedure is illustrated by the same example.

$$[\mathrm{A} \;\vdots\; b] = \begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix}$$

Perform line operations to reach the elimination process:

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix} \begin{matrix} \\ L_2 - (-20/80)L_1 \\ L_3 - (-20/80)L_1 \end{matrix}$$

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ 0 & 35 & -25 & \vdots & 25 \\ 0 & -25 & 125 & \vdots & 25 \end{bmatrix} \begin{matrix} \\ \\ L_3 - (-25/35)L_2 \end{matrix}$$

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ 0 & 35 & -25 & \vdots & 25 \\ 0 & 0 & 750/7 & \vdots & 300/7 \end{bmatrix} \quad \rightarrow \quad \begin{matrix} x_1 = 0.60 \\ x_2 = 1.00 \\ x_3 = 0.40 \end{matrix}$$

The return substitution step is presented next to the triangular matrix at the end of the elimination process.

6. Pivoting

The element of the main diagonal is called the pivot. The method of pivoting is necessary only if the first pivot a_{11} is equal to zero. The procedure also fails if a subsequent pivot $a_{i\,i}$ is null. Although there may be no zeros on the main diagonal of the original matrix, the elimination process can create zeros on the main diagonal. The simple elimination procedure described so far should be modified to avoid zeros on the main diagonal.

This result can be done by rearranging the equations, allowing equations (rows) or variables (columns), before each elimination step to put the element of the highest value on the diagonal. This process is called pivot or failover. By swapping rows and columns is called pivot method. The total pivot is quite complicated, and therefore it is rarely used. By exchanging only the lines one defines partial pivot. Only partial pivoting is considered in this handout.

The pivot method eliminates zeros in pivot locations during the elimination process. This method also reduces rounding errors, as the pivot value is a divisor during the elimination process, and dividing by a large number gives small rounding errors than dividing by a small number. When the procedure is repeated, rounding errors can be serious. This problem becomes more and more serious when the number of equations increases.

Let's use partial pivot elimination to solve the following system of linear equations, Ax = b:

$$\begin{bmatrix} 0 & 2 & 1 \\ 4 & 1 & -1 \\ -2 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

Let's apply the elimination procedure by increasing A with b. The first pivot element is zero, so that pivoting is necessary. The largest value in the first column under the pivot is the second row. Thus, by allowing the first and second row, as well as the new second row already has a zero under the pivot, the elimination process is done just for the third row.

$$\begin{bmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 2 & 1 & \vdots & 5 \\ -2 & 3 & -3 & \vdots & 5 \end{bmatrix} L_3 - (-2/4)L_1$$

When performing elimination operations, one finds:

$$\begin{bmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 2 & 1 & \vdots & 5 \\ 0 & 7/2 & -7/2 & \vdots & 7/2 \end{bmatrix}$$

Although the new pivot in the second row is different from zero, but it is not the largest element of the second column. So a swivel is done again. Note that the swivel is based only on the lines below the swivel. The lines above pivot have already been processed by the elimination process. Using one of the lines above the pivot would destroy the elimination already accomplished. Using the elimination multiplier, evaluate the new values after exchanging the second and third lines.

$$\begin{bmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 7/2 & -7/2 & \vdots & 7/2 \\ 0 & 2 & 1 & \vdots & 5 \end{bmatrix} L_3 - (4/7)L_2$$

Performing elimination operations, the following are:

$$\begin{bmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 7/2 & -7/2 & \vdots & 7/2 \\ 0 & 0 & 3 & \vdots & 3 \end{bmatrix} \rightarrow \begin{matrix} x_1 = -1 \\ x_2 = 2 \\ x_3 = 1 \end{matrix}$$

The results obtained by the return replacement are presented next to the augmented triangular matrix.

7. Scaling

The elimination process described so far can attract significant rounding errors when the values of pivot elements are smaller than the values of other elements in the equations containing pivot elements. In such cases, scaling is used to select pivot elements. After pivoting, elimination is applied to the original equations. Scaling is used only to select pivot elements.

Scale swivel is performed as follows; before applying elimination on the first column, all elements of the first column are scaled (i.e. normalized) by the largest elements in the corresponding rows. Pivoting is implemented based on the scaled elements in the first column, and elimination is applied to obtain null elements in the first column below the pivot element.

Before applying the elimination to the second column, all the elements from 2 to n in the second column are scaled, the pivoting is performed, and the elimination is applied to obtain null elements of the second column below the pivot element. The procedure is applied on the remaining lines from 3 to n - 1. The replacement by return is then applied to obtain x.

Let us look at the advantage of extension (mantissa) by solving the following linear system:

$$\begin{bmatrix} 3 & 2 & 105 \\ 2 & -3 & 103 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 104 \\ 98 \\ 3 \end{bmatrix}$$

Which has exact solution x_1 = -1.0, x_2 = 1.0, et x_3 = 1.0. The effect of rounding can be enhanced by only including three significant figures in our calculations. For the first column, pivoting is not necessary, the matrix A increased and the first set of operations on the rows are given by:

$$\begin{bmatrix} 3 & 2 & 105 & \vdots & 104 \\ 2 & -3 & 103 & \vdots & 98 \\ 1 & 1 & 3 & \vdots & 3 \end{bmatrix} \begin{matrix} \\ L_2 - (0.667)L_1 \\ L_3 - (0.333)L_1 \end{matrix}$$

which gives:

$$\begin{bmatrix} 3 & 2 & 105 & \vdots & 104 \\ 0 & -4.33 & 33.0 & \vdots & 28.6 \\ 0 & 0.334 & -32.0 & \vdots & -31.6 \end{bmatrix} \begin{matrix} \\ \\ L_3 - (-0.077)L_2 \end{matrix}$$

Pivoting is not required for the second column. After the indicated elimination, the triangular matrix is obtained

$$\begin{bmatrix} 3 & 2 & 105 & \vdots & 104 \\ 0 & -4.33 & 33.0 & \vdots & 28.9 \\ 0 & 0 & -29.5 & \vdots & -29.4 \end{bmatrix}$$

When performing the return replacement, we find x_3 = 0.997, x_2 = 0.924, and x_1 = -0.844, which are not in very good agreement with the exact solution x_3 = 1.0, x_2 = 1.0, and x_1 = -1.0. Rounding errors due to three-digit accuracy disrupted the accuracy of the solution. The effects of rounding can be reduced by scaling equations before pivoting. Scaling should be used only to determine if pivoting is necessary. All calculations must be done before scaling.
Let's try to take the last system that requires scaling to determine if a swivel is required. The first step in the disposal procedure is to remove all items from the first column under item a_{11}. To make this step more efficient, divide all elements of the first column by the largest element in each row. The result is:

$$a_1 = \begin{bmatrix} 3/105 \\ 2/103 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0.0286 \\ 0.0194 \\ 0.3333 \end{bmatrix}$$

With a_1 is the notation of the vector that contains the division of the elements of the first column of the matrix A. The third element of the vector a_1 is the largest of the elements in a_1, a change between the first row and the third row is necessary. The elimination operations indicated are as follows:

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 3 \\ 2 & -3 & 103 & \vdots & 98 \\ 3 & 2 & 105 & \vdots & 104 \end{bmatrix} \begin{matrix} \\ L_2 - (2/1)L_1 \\ L_3 - (3/1)L_1 \end{matrix}$$

The results obtained are indicated as follows with the new disposal operation:

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 3 \\ 0 & -5 & 97 & \vdots & 92 \\ 0 & -1 & 96 & \vdots & 95 \end{bmatrix} \begin{matrix} \\ \\ L_3 - (1/5)L_2 \end{matrix}$$

Dividing the second and third elements of the second column as follows:

$$a_2 = \begin{bmatrix} -5/97 \\ -1/96 \end{bmatrix} = \begin{bmatrix} -0.0516 \\ -0.0104 \end{bmatrix}$$

Therefore, the pivoting is not indicated, the results are:

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 3 \\ 0 & -5 & 97 & \vdots & 92 \\ 0 & 0 & 76.6 & \vdots & 76.6 \end{bmatrix}$$

Thus by proceeding to the replacement by return, we find x_1 = 1.00, x_2 = 1.00 and x_3 = -1.00, which corresponds to the exact solution. Following this procedure, the effect of rounding was avoided.

8. Gauss elimination

The elimination procedure described in the previous section, including swivel scaling, is commonly referred to as Gauss elimination. It is the most important and useful direct elimination method for solving systems of algebraic linear equations. The Gauss-Jordan method, the inverse matrix method and the LU factorization method are all modifications or extensions of the Gauss elimination method. The pivot is an essential element of Gauss elimination. In cases where all elements of the A matrix are of the same order of magnitude, scaling is not required. However, the pivoting rule to avoid zero in pivot is necessary in an elimination procedure.

The pivoting scaling method is used to reduce rounding errors, although highly desirable in general, it may be subject to a risk to the accuracy of the solution. When removing Gauss by hand, decisions about pivoting can be made on a case-by-case basis. When writing a computer program of the general-purpose Gauss elimination method for systems of equations, however, the method of pivoting scaling is an absolute necessity.

The Gauss elimination procedure, in a suitable format for computer programming, is summarized as follows:

1. Define an nxn size matrix consisting of the coefficients of the linear equation system.
2. From the first column, we look for the largest element in this column and we put this coefficient in the pivot position (pivot or permutation of rows).
3. For column k (k = 1, 2 ..... n - 1), we apply the procedure of elimination of rows i (i = k + 1, k + 2 ..... n) in order to create zeros in the column below the pivot $a_{k,k}$, so that our matrix turns into a triangular matrix as follows:

$$a_{i,j} = a_{i,j} - \left(\frac{a_{i,k}}{a_{k,k}}\right) a_{k,j} \quad (i,j=k+1,\ k+2,\dots,n)$$

$$b_i = b_i - \left(\frac{a_{i,k}}{a_{k,k}}\right) b_k \quad (i=k+1,\ k+2,\dots,n)$$

4. After step 3 was applied to all columns of k, (k = 1.2 ..... n - 1), the input matrix A the original became a superior triangular matrix. The resolution of our system is done by a feedback. Thus we have:

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j} x_j}{a_{i,i}} \qquad (i = n-1, n-2, \dots, 1)$$

9. Gauss-Jordan Elimination

Gauss-Jordan elimination is a variant of Gauss elimination, in which the elements above the main diagonal are eliminated (make them null) as well as the elements below the main diagonal, to convert the matrix *A* to a diagonal matrix. The lines are scaled to make the diagonal elements equal to 1, which transforms the matrix *A* to an identity matrix. Vector b is then transformed into solution vector *x*. The number of multiplications and divisions for Gauss-Jordan elimination is about 50 percent larger than for Gauss elimination. Therefore, the elimination of Gauss is preferable.

Let's work on the same example discussed before, this example does not require a pivoting, so the augmented matrix is:

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix} \begin{array}{l} L_1/80 \\ \\ \\ \end{array}$$

The division of the first line by 80 is carried out to make the pivot a_{11}=1

$$\begin{bmatrix} 1 & -1/4 & -1/4 & \vdots & 1/4 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix} \begin{array}{l} \\ L_2-(-20)L_1 \\ L_3-(-20)L_1 \end{array}$$

Applying the deletions of the two elements below the pivot with the following changes on the second and third row:

$$\begin{bmatrix} 1 & -1/4 & -1/4 & \vdots & 1/4 \\ 0 & 35 & -25 & \vdots & 25 \\ 0 & -25 & 125 & \vdots & 25 \end{bmatrix} \begin{array}{l} \\ L_2/35 \\ \\ \end{array}$$

Let’s divide the second line by 35 to make the new pivot a_{22}=1

$$\begin{bmatrix} 1 & -1/4 & -1/4 & \vdots & 1/4 \\ 0 & 1 & -5/7 & \vdots & 5/7 \\ 0 & -25 & 125 & \vdots & 25 \end{bmatrix} \begin{array}{l} L_1-(-1/4)L_2 \\ \\ L_3-(-25)L_2 \end{array}$$

Let’s apply the elimination both below and above the second line:

$$\begin{bmatrix} 1 & 0 & -3/7 & \vdots & 3/7 \\ 0 & 1 & -5/7 & \vdots & 5/7 \\ 0 & 0 & 750/7 & \vdots & 300/7 \end{bmatrix} \begin{array}{l} \\ \\ L_3/(750/7) \end{array}$$

Let’s divide the third line by 750/7 to make the last pivot a_{33}=1

$$\begin{bmatrix} 1 & 0 & -3/7 & \vdots & 3/7 \\ 0 & 1 & -5/7 & \vdots & 5/7 \\ 0 & 0 & 1 & \vdots & 2/5 \end{bmatrix} \begin{array}{l} L_1-(-3/7)L_3 \\ L_2-(-5/7)L_3 \\ \\ \end{array}$$

Applying the elimination above the third line results in:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 0.60 \\ 0 & 1 & 0 & \vdots & 1.00 \\ 0 & 0 & 1 & \vdots & 0.40 \end{bmatrix}$$

The matrix *A* thus became an identity matrix and the vector b became a solution vector: X = [0.60 1.00   0.40].

10. Matrix Inversion

The inverse of a square matrix A is the matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Gauss-Jordan elimination can be used to determine the inverse of a matrix A by increasing A with the matrix identity I. Applying the Gauss-Jordan algorithm, the matrix A is transformed to an identity matrix I, and the identity matrix is transformed to an inverse matrix, A^{-1}. Thus, the application of the Gauss-Jordan elimination allows writing:

L'inverse d'une matrice carrée A est la matrice A^{-1} tel que $AA^{-1} = A^{-1}A = I$. L'élimination de Gauss-Jordan peut être utilisée pour déterminer l'inverse d'une matrice A en augmentant A avec la matrice identité I. En appliquant l'algorithme de Gauss-Jordan, la matrice A est transformée à une matrice d'identité I, et la matrice d'identité est transformée en matrice inverse, A^{-1}. Using the Gauss-Jordan elimination application, you can write:

$$[A \quad \vdots \quad I] \rightarrow [I \quad \vdots \quad A^{-1}]$$

The Gauss-Jordan elimination procedure, in a suitable format for computer programming, can be developed, modifying the first step. Increase the matrix A of nxn by combining the matrix identity I of nxn. Steps 2 and 3 of the method are the same, adapting the pivot to the unit by dividing all the elements in the row by the value of the pivot. The elimination is carried out above, as well as below the pivot. At the end, matrix A will be transformed into identity matrix, and the input identity matrix will be transformed into inverse matrix A^{-1}.
Let's evaluate the inverse matrix of matrix A used in the Gauss-Jordan elimination section. At the beginning, increase the matrix A by the matrix identity I, as follows:

$$[A \quad \vdots \quad I] = \begin{bmatrix} 80 & -20 & -20 & \vdots & 1 & 0 & 0 \\ -20 & 40 & -20 & \vdots & 0 & 1 & 0 \\ -20 & -20 & 130 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Using the Gauss-Jordan elimination method, the augmented matrix is transformed into the following augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 2/125 & 1/100 & 1/250 \\ 0 & 1 & 0 & \vdots & 1/100 & 1/30 & 1/150 \\ 0 & 0 & 1 & \vdots & 1/250 & 1/150 & 7/750 \end{bmatrix} = [I \quad \vdots \quad A^{-1}]$$

So the inverse matrix of A is:

$$A^{-1} = \begin{bmatrix} 2/125 & 1/100 & 1/250 \\ 1/100 & 1/30 & 1/150 \\ 1/250 & 1/150 & 7/750 \end{bmatrix}$$

If we multiply the matrix A with the inverse matrix A^{-1} we find the identity matrix. An equation system can be solved using the matrix inversion method A^{-1}. A linear equation system is written:

$$A\ x = b$$

By multiplying the two parts on the left by A^{-1}, we have:

$$A^{-1} A\ x = A^{-1} b$$

$$I\ x = A^{-1} b$$

This allows us to:

$$x = A^{-1} b$$

Thus, when the inverse matrix A^{-1} of the matrix A of the equation system is known, the solution (the vector x) is simply the product of the inverse matrix A^{-1} by the vector b. The square matrixes are not all invertible. The singular matrix is a matrix whose determinant is zero, does not have an inverse matrix. The corresponding system of equations has no single solution.

Using the inverse matrix method, our equation system is solved by multiplying the inverse matrix obtained by vector b as follows:

$$X = A^{-1}b = \begin{bmatrix} 2/125 & 1/100 & 1/250 \\ 1/100 & 1/30 & 1/150 \\ 1/250 & 1/150 & 7/750 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

This results in:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (2/125)(20) & + & (1/100)20) & + & (1/250)(20) \\ (1/100)(20) & + & (1/30)(20) + & & (1/150)(20) \\ (1/250)(20) & + & (1/150)(20) + & & (7/750)(20) \end{bmatrix} = \begin{bmatrix} 0.60 \\ 1.00 \\ 0.40 \end{bmatrix}$$

11. Factorization Method

A matrix (such as a scalar) can be factored into two other matrices so that their product gives the original matrix as follows:

A = BC

When B and C are lower and upper triangular matrices, respectively, the expression A becomes

A = LU

By specifying the diagonal elements of L or U, the factorization becomes unique. Two processes exist, the first considers that the elements of the main diagonal of L are all equal to one (the process is called the Doolittle method). The second considers that the elements of the main diagonal of U equal to one (the process is called the Crout Method).

The factorization method of a matrix is used to reduce the work required for the Gauss elimination method when several unknown b vectors are to be considered. The Doolittle LU process is accomplished by defining 'em' elimination multipliers determined in the elimination step of the Gauss elimination method as elements of the L matrix. The U matrix is defined as the upper triangular matrix determined by the elimination step of the Gauss elimination method. In this way, multiple b vectors can be processed through the elimination step using the L matrix and through the return substitution step using elements of the U matrix.

Consider the linear system Ax = b, either the matrix A factored into the product of two matrices LU, the linear system becomes:

LUx = b

Multiplying by L^{-1} equality becomes:

$L^{-1}LUx = L^{-1}b$

$Ux = L^{-1}b$

Posing a new vector b' such as:

$b' = L^{-1}b$

$Ux = b'$

Multiplying by L, equality becomes:

$Lb' = LL^{-1}b$

$Lb' = b$

To solve our starting system Ax = b, we must solve both subsystems: the first Lb' = b, knowing that L is a lower triangular matrix. By substitution by return the elements of the vector b' will be determined, the system unknowns can be easily calculated with the second system Ux = b'. Solving our system of the previous example will allow us to better understand the factorization method.

Or the following system of linear equations:

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix}$$

With factorization, our initial matrix A is transformed into two matrices as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \quad et \quad U = \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix}$$

To begin with, b' is determined by the resolution of:

$Lb' = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

We find:

$$\begin{bmatrix} b'_1 = 20 \\ b'_2 = 20 - (-1/4)(20) = 25 \\ b'_3 = 20 - (-1/4)(20) - (-5/7)(25) = 300/7 \end{bmatrix}$$

Now, with the vector b' the unknowns of the system Ux = b' are determined as follows:

$$\begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 300/7 \end{bmatrix}$$

By replacing the back, the vector x is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 1 \\ 0.40 \end{bmatrix}$$

The LU factorization method, in an appropriate format for computer programming, is summarized as follows:

1. Perform the same steps in the Gauss Removal Method described in the previous section. Store swivel information in an order vector. Store the “em” elimination multipliers at the location of the eliminated items. The results of this step are the L and U matrices.
2. Calculate the vector b' in the order of the elements of the control vector using the substitution before:

$$b_i' = b_i - \sum_{k=1}^{i-1} l_{i\,k} b_k' \qquad (i = 2, 3, \dots, n)$$

With $l_{i\,k}$ are the matrix elements L.

3. Calculate vector x using the back substitution:

$$x_i = b_i' - \sum_{k=i+1}^{n} u_{i\,k} x_k \,/ u_{i\,i} \qquad (i = n-1, n-2, \dots, 1)$$

With $u_{i\,k}$ and $u_{i\,i}$ are the elements of the U matrix.

Finally, the LU factorization method can be used to determine the inverse matrix. The inverse matrix is calculated column by column to use the unit vector instead of b to solve Lb' = b as follows:

For comparison purposes, the same matrix is used:

$$\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix}$$

After factorization, we have:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \quad et \quad U = \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix}$$

First determine b_1' by the resolution of $Lb_1' = b_1$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad b_1' = \begin{bmatrix} 1 \\ 1/4 \\ 3/7 \end{bmatrix}$$

Calculate $\mathrm{Ux} = b_1'$ to determine x1 as follows:

$$\begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 3/7 \end{bmatrix} \quad \rightarrow \quad x_1 = \begin{bmatrix} 2/125 \\ 1/100 \\ 1/250 \end{bmatrix}$$

The result obtained which is vector x_1, represents the first column of the matrix A^{-1}. Identically, with $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ give $x_2 = \begin{bmatrix} 1/100 \\ 1/30 \\ 1/150 \end{bmatrix}$ and $b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ give $x_3 = \begin{bmatrix} 1/250 \\ 1/150 \\ 7/750 \end{bmatrix}$

In the end, A^{-1} is:

$$A^{-1} = [x_1 \quad x_2 \quad x_3] = \begin{bmatrix} 2/125 & 1/100 & 1/250 \\ 1/100 & 1/30 & 1/150 \\ 1/250 & 1/150 & 7/750 \end{bmatrix}$$

This is the same result obtained by the Gauss-Jordan elimination method.

VII. Reference

[1] François Liret & Dominique Martinais, « Cours de mathématiques, Algèbre 1ère année », 2003, Dunod.

[2] Joe D. Hoffman, « Numerical Methods for Engineers and Scientists », New York, 1992, Marcel Dekker, Inc.