International Journal of Geometric Methods in Modern Physics
Vol. 10, No. 4 (2013) 1350001 (22 pages)

DOI: 10.1142/S0219887813500011

MULTISYMPLECTIC GEOMETRY AND k-COSYMPLECTIC STRUCTURE FOR THE FIELD THEORIES AND THE RELATIVISTIC MECHANICS

H. LOUMI-FERGANE

Département de Physique, Faculté des Sciences
Université Es-Senia, Oran 31000, Algerie
h_loumi@yahoo.com

A. BELAIDI

Département de Génie Electrique, ENSET d'Oran
Oran 31000, Algerie
belaidia@yahoo.fr

Received 16 January 2012
Accepted 17 September 2012
Published 15 January 2013

The aim of this work is twofold: First, we extend the multisymplectic geometry already done for field theories to the relativistic mechanics by introducing an appropriate configuration bundle. In particular, we developed the model to obtain the Hamilton–De Donder–Weyl equations to the movement of a relativistic charged particle immerged in an electromagnetic field. Second, we have found a direct relationship between the multisymplectic geometry and the k-cosymplectic structure of a physical system.

Keywords: Field theories; relativistic and classical mechanics; Lagrangian formalism; Hamiltonian formalism; k-symplectic structure; k-cosymplectic structure; multisymplectic geometry.

1. Introduction

In recent years much work has been done with the aim of establishing the suitable geometrical models for making a covariant description for both models: the field theories and the relativistic mechanics.

There are different kinds of geometrical models. We have the so-called k-symplectic formalism which uses the k-symplectic forms introduced by Awane [1–3], and which coincides with the polysymplectic formalism described by Günther [4]. From this geometrical model many of the characteristics of the autonomous Hamiltonian systems arise from the existence of a natural geometric structure with which the phase space of the systems is endowed by the symplectic form (a closed and one-nondegenerate two-form). A natural extension of this is the k-cosymplectic formalism which is the generalization to field theories of the cosymplectic ($k = 1$) description of non-autonomous mechanical systems [5, 6]. On linear frame bundle, soldering forms are also polysymplectic forms where their study and applications to field theories constitute the k-symplectic geometry developed by Norris [7–9].

Among the different geometrical models cited above for describing field theories, there are multisymplectic models [10–13], first introduced by Kijowski and Tulczyjew [14–16].

On this model, the multisymplectic manifold is endowed with a closed and one-nondegenerate k-form. In this study, we are interested on the Lagrangian and Hamiltonian formalisms.

The jet bundles $J^1\pi$ are the appropriate domain for the construction of the Lagrangian formalism [17–22]. In particular, this formalism is constructed for (hyper)-regular and singular field theories Roman-Roy [23].

The Hamiltonian description presents different kinds of problems because the choice of the multimomentum bundle for developing the theory is not unique [24] and different kinds of Hamiltonian systems can be defined depending on this choice and on the way of introducing the physical content the "Hamiltonian" [25–30]. To overcome this difficulty [25], all kinds of multimomentum bundle are subbundle of a larger one which is the extended multimomentum bundle. In the Hamiltonian formalism, the standard way of defining Hamiltonian systems is based on using Hamiltonian sections [31] and which are the Hamiltonian counterparts of Lagrangian systems. The construction of these Hamiltonian systems is carried out by using the Legendre map associated with the Lagrangian system. This problem is particularly studied for the (hyper)-regular case [22, 32] which is developed in the restricted multimomentum bundle and for the singular (almost-regular) case [21, 33, 34] is not defined everywhere in the restricted or the extended multimomentum bundles, but it is developed in certain submanifolds in one or the other of them. These constructions are reviewed in [23].

In the geometric description of field theories, the most interesting subject concerns the field equations. In the multisymplectic models, both in the Lagrangian and Hamiltonian formalisms, these equations can be derived from a suitable variation principle: the so-called Hamilton principle in the Lagrangian formalism and Hamilton–Jacobi principle in the Hamiltonian formalism [17, 19, 24, 25]. Particularly, for the hyper-regular theories, the variation principle leads to Hamilton–De Donder–Weyl (HDW) equations by using special kinds of integrable multivector fields (holonome multivector fields) [23, 25], which make these equations in a suitable geometric formulation [35].

The purpose of this paper is to extend the multisymplectic geometry already done for the field theories to the relativistic mechanics. Using this extension, we can define new kinds of holonome multivector fields which verify the Hamilton principle. In particular, in Hamiltonian formalism, we obtained the HDW equations describing the movement in mechanics.

The paper is organized as follows: First, we summarize the multisymplectic geometry field theories, in particular in our work, we pointed out that the hyper-regular field theories can be developed on a configuration whose base is identified with the flat space: Minkowski space. All these aspects are discussed in Sec. 2. Then, we develop our model of multisymplectic geometry for relativistic mechanics in Sec. 3. Section 4 is devoted to retrieve a relationship between the symplectic two-form ω_L of the k-structure (k-symplectic and k-cosymplectic) and the Poincaré–Cartan form Ω_L of the multisymplectic geometry. Finally, in Sec. 5, we develop the multisymplectic model given above in Secs. 2 and 3 separately to study the dynamic of a free electromagnetic field and the movement of a charged particle immerged in an electromagnetic field A_μ respectively.

2. Multisymplectic Geometry for Classical Field Theories

2.1. *Lagrangian formalism*

The field theories are the classical limit of quantum fields' theories. Those are the fields, such as gauge fields of Yang–Mills, which interact with matter fields. A geometric description has already been done [36] in building a principal fiber bundle $G \times S^{0,2}$ where $G \equiv$ Lie group associated in this case to the quantum fields of Yang–Mills. This fiber is above a database the flat space: Minkowski space ($k = 4$) which coincides with the form of the Lagrangian of fields that we studied (i.e. Lagrangian which is only explicit on fields not on the database coordinates $(x^\mu)_{\mu=\overline{0,3}}$). The classical limit of these Lagrangians corresponds to the study of fields without constraints (this coincides with the abstraction of ghosts which corresponds to the $S^{0,2}$ group). The favorable principal fiber of configuration is $E = G(G \equiv N)$ and the structure in this case is four-symplectic (i.e. $L_0 \in (T^1_4 N)$). How to use the multisymplectic geometry of such theories? This problem is solved in Sec. 4.

In this section, we are going to summarize the multisymplectic geometry given for studying the dynamic of field theories [23, 25]. In particular, we are going to concentrate ourselves on dynamic of most general case of field theories: theories whose Lagrangians are explicit on database coordinates $(x^\mu)_{\mu=\overline{0,3}} = (x^0 = ct, (x^i)_{i=\overline{1,3}})$ and are hyper-regular. So, we follow the following steps.

Let $\pi : E \to M$ be a fiber bundle with M the base space, that we choose, is a flat manifold i.e. the Minkowski space with global coordinates $\{x^\mu\}$. π is the pull-back of a section $\phi : M \to E$

$$x^\mu \to (x^\mu, y^A = \phi^A(x^\rho)); \quad \mu = 0, \dots; \quad A = 1, \dots, d,$$

where $\{y^A\} \equiv$ physical fields. These fields $\{y^A\}$ are presented by a fiber above each (x^μ) of the base space M. The set of fibers is denoted by the space N so the fiber bundle E will be

$$E = R^4 \times N. \tag{1}$$

Let $\pi^1 : J^1\pi \to E$ be the first-order jet bundle of π. By using (1),

$$J^1\pi = R^4 \times T_4^1 N, \tag{2}$$

where $T_4^1 N$ is the Whitney sum of four-copies of the tangent space TN at the space N with local coordinates $(y^A, v_v^A), \dim T_4^1 N = 5d$. π^1 is the pullback of a section which is a mapping $\psi : E \to J^1\pi$. If ψ is a global section of π^1 such that $\pi^1 \circ \psi = \mathrm{Id}_E$, ψ is called a jet field. In this case ϕ is an integral section of ψ and $\psi \circ \phi = j^1\phi$ (where $j^1\phi : M \to J^1\pi$ denotes the canonical lifting of ϕ) and ψ is the integral jet field.

If (x_v) is a natural local system on $M, (x_v, y^A, v_v^A)$ is the induced local coordinates system on $J^1\pi$ where

$$j^1\phi(x_v) = (x_v, y^A, v_v^A) = (x_v, \phi^A(x_\rho),\ \partial_v\phi^A(x_\rho)), \tag{3}$$

with

$$\partial_v\phi^A = \frac{\partial\phi^A}{\partial x^v} = v_v^A \equiv \text{velocity of field.}$$

Let $\overline{\pi}^1 \equiv \pi \circ \pi^1$: $J^1\pi \to M$, where $\overline{\pi}^1$ is the pullback of the section $J^1\phi$.

A Lagrangian density is usually written as $\mathrm{L} = L(\overline{\pi}^{1*}\eta)$ where $L \in C^\infty(J^1\pi)$ is the Lagrangian function and η is the volume form on $M(\eta \in \Omega^4(M))$ with

$$\eta = dx^k = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad k = 4.$$

By using the natural system of coordinates defined on $J^1\pi$, the expression of the Lagrangian density is:

$$\mathrm{L} = L(x^\mu, y^A, v_\mu^A)dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{4}$$

And the expressions of θ_L and Ω_L the Poincaré–Cartan four- and five-forms are respectively [23]:

$$\theta_L = \frac{\partial L}{\partial v_v^A} dy^A \wedge d^{k-1}x_v - \left(\frac{\partial L}{\partial v_v^A} v_v^A - L\right) d^k x \quad \text{and} \quad \Omega_L = -d\theta_L \in \Omega^5(J^1\pi), \tag{5}$$

where $d^{k-1}x_\alpha \equiv i(\frac{\partial}{\partial x^\alpha})d^k x$.

Let $\Gamma(M, E)$ be the set of sections $\{\phi\}$ cited above and $(J^1\pi, \Omega_L)$ be the Lagrangian system. The Lagrangian field equations can be derived from a variational principle called the Hamilton principle associated to the Lagrangian formalism which is given by:

$$i(X_L)\Omega_L = 0, \tag{6}$$

where $\{X_L\} \subset \chi^4_L(J^1\pi)$ is a class of holonomic multivector fields associated to $j^1\phi$ (X_L is $\overline{\pi}^1$-transverse, integrable and SOPDE). The local expression of X_L is given by:

$$X_L = \frac{\partial}{\partial x^v} + F^A_v \frac{\partial}{\partial y^A} + G^A_{v\rho} \frac{\partial}{\partial v^A_\rho}, \tag{7}$$

where $F^A_v = v^A_v$ and $G^A_{v\rho} = \frac{\partial^2 y^A}{\partial x^v \partial x^\rho}$.

By substituting (7) and (5) in (6), the Euler–Lagrange equations for the fields satisfy:

$$\left(\frac{\partial L}{\partial y^A} - \frac{\partial}{\partial x^v} \left(\frac{\partial L}{\partial v^A_v} \right) \right) \circ j^1\phi = 0 \quad \forall\, A = \overline{1, d}. \tag{8}$$

In this case $\frac{\partial^2 L}{\partial v^A_\mu \partial v^B_v} \neq 0$, $\forall\, (\overline{y}) \in J^1\pi$, the Lagrangian is hyper-regular (regular globally), so we can associate a Hamiltonian formalism equivalent to the Lagrangian one.

2.2. *Hamiltonian formalism*

At this $(J^1\pi, \Omega_L)$ the associated extended Hamiltonian system is $(M\pi, \Omega)$ where $M\pi$ is a subbundle of the multicotangent bundle of E (i.e. $M\pi \subset \bigwedge^4 T^*E$ the total exterior algebra bundle) $M\pi \equiv \bigwedge^4_2 T^*E$, $\dim M\pi = d + 4 + 4d + 1 = 5d + 5$ [25].

Let θ and Ω, the multimomentum Liouville four- and five-forms defined on $M\pi$. If the local coordinates (x^v, y^A, p^v_A, p) in $M\pi$, the local expressions of these forms are:

$$\begin{aligned} \theta &= p^v_A dy^A \wedge d^{k-1}x_v - p d^k x, \\ \Omega &= -d\theta = -dp^v_A \wedge dy^A \wedge d^{k-1}x_v + dp \wedge d^k x. \end{aligned} \tag{9}$$

Let $\tilde{F}L$ be the extended Legendre map $\tilde{F}L : J^1\pi \to M\pi$. In natural coordinates:

$$\begin{aligned} \tilde{F}L^* x^v = x^v, \quad \tilde{F}L^* y^A = y^A, \quad \tilde{F}L^* p^A_v = \frac{\partial L}{\partial v^A_v}, \\ \tilde{F}L^* p = v^A_v \frac{\partial L}{\partial v^A_v} - L, \quad \tilde{F}L^*\theta = \theta_L \quad \text{and} \quad \tilde{F}L^*\Omega = \Omega_L. \end{aligned} \tag{10}$$

Because $(J^1\pi, \Omega_L)$ is hyper-regular, it is equivalent to a Hamiltonian system $(J^1\pi^*, \Omega_h)$ by a global diffeomorphism FL called the restricted Legendre map associated at L. Let $J^1\pi^*$ be the restricted multimomentum bundle associated at $\pi : E \to M$, so, $J^1\pi^* = R^4 \times (T^1_4)^*N$ where $(T^1_4)^*N$ is the Whitney sum of four-copies of the cotangent space T^*N at N spawned by (y^A, p_v^A) and $\dim J^1\pi^* = 5d + 4(\dim\,(T_4^1)^*N = 5d)$.

Let the natural submersion $\mu : M\pi \to J^1\pi^*$ and $FL := \mu \circ \tilde{FL} : J^1\pi \to J^1\pi^*$. The coordinates of both bundles are related by:

$$FL^* x^v = x^v, \quad FL^* y^A = y^A, \quad FL^* p_v^A = \frac{\partial L}{\partial v_v^A}, \tag{11}$$

where (x_v, y^A, p_v^A) is a natural coordinate system defined on $J^1\pi^*$.

To the projection μ, we associate a section $h : J^1\pi^* \to M\pi$. It is called a Hamiltonian section which carries the physical information of the system. We can do the following remark that, in the case of free fields, the Hamiltonian section is specified by a Hamiltonian function $h \in C^\infty(J^1\pi^*)$ (i.e. h is defined globally on $J^1\pi^*$), thus the map $h = \mu^{-1} := \tilde{FL} \circ FL^{-1}$. Locally, the Hamiltonian section $h(x^v, y^A, p_A^v) = (x^v, y^A, p_A^v, p = -h(x^\rho, y^B, p_B^\rho))$ is specified by the Hamiltonian function

$$h = (FL^{-1})^* L - p_A^v (FL^{-1})^* v_v^A. \tag{12}$$

On $J^1\pi^*$, the local expressions of the Hamilton–Cartan four- and five-forms are defined by:

$$\begin{aligned} \theta_h &= p_A^v dy^A \wedge d^{k-1}x_v + hd^k x, \\ \Omega_h &= -d\theta_h = -dp_A^v \wedge dy^A \wedge d^{k-1}x_v - dh \wedge d^k x. \end{aligned} \tag{13}$$

By substituting (11) in (13), we obtain (5):

$$FL^*\theta_h = \theta_L, \quad FL^*\Omega_h = \Omega_L.$$

In order to obtain the Hamiltonian fields' equations equivalent to those obtained by the Lagrangian formalism (8), we use the Hamilton–Jacobi principal:

$$i(X_h)\Omega_h = 0, \tag{14}$$

where X_h is the HDW multivector field (i.e. HDW multivector field is a class of integrable and $\overline{\tau}$-transverse the multivector fields $\{X_h\} \subset \chi^4(J^1\pi^*)$):

$$X_h = \frac{\partial}{\partial x^v} + F_v^A \frac{\partial}{\partial y^A} + G_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} \tag{15}$$

with

$$F_v^A = \frac{\partial h}{\partial p_A^v}, \quad G_{A\nu}^v = -\frac{\partial h}{\partial y^A}$$

and $\overline{\tau} : J^1\pi^* \to M$ pullback of an integral section Ψ associated at ϕ such that:

$$FL \circ J^1\phi := \Psi : M \to J^1\pi^*. \tag{16}$$

Finally, in a natural system of coordinates (x^v, y^A, p^v_A) in $J^1\pi^*$, Ψ satisfies the following system of equations:

$$\begin{aligned}\frac{\partial(y^A\circ\Psi)}{\partial x^v} &= \frac{\partial h}{\partial p^v_A}\circ\Psi \equiv \left.\frac{\partial h}{\partial p^v_A}\right|_{\Psi},\\ \frac{\partial(p^v_A\circ\Psi)}{\partial x^v} &= -\frac{\partial h}{\partial y^A}\circ\Psi \equiv \left.\frac{\partial h}{\partial y^A}\right|_{\Psi}.\end{aligned} \tag{17}$$

These equations are known as the HDW equations of the restricted Hamiltonian system.

We define the vector $Z \equiv (y^A, p^\mu_A)^T$ and $\nabla_Z \equiv (\partial_{y^A}, \partial_{p^\mu_A})$ for $A = \overline{1,d}$ and $\mu = \overline{0,3}$ where both the components $\{y^A\}$ and $\{p^\mu_A\}$ are unconstrained in $J^1\pi^*$. Having regards to the choice of the configuration bundle E and the Lagrangian of fields, if we explicit the Hamiltonian equation fields (17), we obtain:

$$\begin{pmatrix}\frac{\partial h}{\partial y^A}\\ \frac{\partial h}{\partial p^0_A}\\ \frac{\partial h}{\partial p^1_A}\\ \frac{\partial h}{\partial p^2_A}\\ \frac{\partial h}{\partial p^3_A}\end{pmatrix} = \underbrace{\begin{pmatrix}0&-1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}}_{J^0}\partial_0\begin{pmatrix}y^A\\p^0_A\\p^1_A\\p^2_A\\p^3_A\end{pmatrix} + \underbrace{\begin{pmatrix}0&0&-1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}}_{J^1}\partial_1\begin{pmatrix}y^A\\p^0_A\\p^1_A\\p^2_A\\p^3_A\end{pmatrix}$$
$$+ \underbrace{\begin{pmatrix}0&0&0&-1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}}_{J^2}\partial_2\begin{pmatrix}y^A\\p^0_A\\p^1_A\\p^2_A\\p^3_A\end{pmatrix} + \underbrace{\begin{pmatrix}0&0&0&0&-1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}}_{J^3}\partial_3\begin{pmatrix}y^A\\p^0_A\\p^1_A\\p^2_A\\p^3_A\end{pmatrix}. \tag{18}$$

The system (18) can be contracted naturally [35] in:

$$\underbrace{(J^\mu\partial_\mu)}_{J_\partial}Z = \nabla_Z h. \tag{19}$$

Equation (19) is the contracted form of the HDW equations for fields whose components are independents (i.e. unconstrained).

The explicit matrices $\{J^\mu\}_{\mu=\overline{0,3}}$ [37] associated to the operator J_∂ are obtained naturally, where J^μ are 5×5 skew-symmetric constant matrix for "A" fixed. The partial differential ∂_μ appearing in J_∂ is associated to the coordinates (x_μ) of the flat space: Minkowski space. The system (19) can also be written as:

$$\nabla_Z h = -i\underbrace{(iJ^\mu \partial_\mu)}_{X} Z = (e^{-iX} - 1)Z. \tag{20}$$

3. Multisymplectic Geometry for the Relativistic Mechanics

3.1. *Lagrangian formalism*

By analogy with the work already done for the field theories, we extend the idea to the relativistic mechanics.

Let $\pi : E = R \times N \to R$, where E is the configuration bundle, R as a base space spawned by (ct) as global coordinate and $N = R^3$ is the fiber above each point of the database ($\dim N = 3$ and $\dim E = 4$). Let $(q^\mu)_{\mu=\overline{0,3}} = (q^0 = ct, (q^i)_{i=\overline{1,3}})$ be a natural coordinate defined in E. If the configuration bundle E can be equipped with a metric $\eta^{\mu\nu} = (1, -1, -1, -1)$ such that $q^\mu = \eta^{\mu\nu} q_v$, in this case E coincides with the Minkowski space. We note that "c" is speed of light and $(q^i)_{i=\overline{1,3}}$ are the generalized coordinates. $\pi^1 : J^1\pi \to E$ is the first-order jet bundle of π. In this case, the section $j^1\phi : R \to J^1\pi := R \times TN$ and where $\dim J^1\pi = 7$.

The natural coordinates defined on $J^1\pi$ as done in (3) is $(q^0, q^i, \dot{q}^i)$ and the global integral section $j^1\phi$ such that:

$$j^1\phi(q^0) = \left(q^0, \phi^i(q^0) = \phi^i(t) = q^i(t), \frac{\partial \phi^i}{\partial q^0}(q^0) = \frac{\partial \phi^i}{c\partial t}(t) = \frac{dq^i}{cdt}(t) = \frac{\dot{q}^i(t)}{c} = \bar{\dot{q}}^i(t)\right). \tag{21}$$

We define the Lagrange function $L : R \times TN \to R$. We define on $J^1\pi$, the Poincaré–Cartan one-form θ_L and two-form Ω_L associated at L as in (5) by:

$$\begin{aligned} \theta_L &= \frac{\partial L}{\partial \dot{q}^i} dq^i - \frac{1}{c}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) dq^0, \\ \Omega_L &= -d\theta_L. \end{aligned} \tag{22}$$

We put

$$\frac{dq^i}{cdt}(t) = \bar{\dot{q}}^i(t) \quad \text{and} \quad \frac{d^2q^i}{cdt^2}(t) = \frac{\ddot{q}^i(t)}{c} = \bar{\ddot{q}}^i(t),$$

where $\dot{q}^i(t)$ and $\ddot{q}^i(t)$ are the velocity and the acceleration of the mechanical system respectively. For the relativistic mechanics, at the Hamilton principal (6), we can associate the following holonomic multivector field, (7) becomes:

$$\overline{X}_L = \frac{\partial}{\partial q^0} + \bar{\dot{q}}^i \frac{\partial}{\partial q^i} + \bar{\ddot{q}}^i \frac{\partial}{\partial \dot{q}^i}. \tag{23}$$

We can do the following remark that

$$\overline{X}_L = \frac{1}{c}\left(\frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \ddot{q}^i\frac{\partial}{\partial \dot{q}^i}\right) = \frac{1}{c}X_L.$$

For this dynamic, the first-order jet bundle $J^1\pi$ is generated by the HDW multivector field $\overline{X}_L$ (i.e. HDW multivector field is a class of integrable) and $\overline{\tau}$-transverse multivector fields $\{\overline{X}_L\} \subset \chi^1(J^1\pi)$. The Lagrangian is hyper-regular, so $(J^1\pi, \Omega_L)$ is equivalent to a Hamiltonian system $(J^1\pi^*, \Omega_h)$.

3.2. *Hamiltonian formalism*

Let $M\pi = \bigwedge_2^1 T^*E \equiv R \times R \times T^*N$ be the extended multimomentum bundle, $\dim M\pi = 8$. The multimomentum Liouville forms, in a natural coordinates (q^0, q^i, p^i, p) defined in $M\pi$ are:

$$\begin{aligned}\theta &= p_i dq^i - p dq^0,\\ \Omega &= -d\theta = -dp_i \wedge dq^i + dp \wedge dq^0.\end{aligned} \tag{24}$$

The first-order jet bundle associated at E is $J^1\pi^* := R\times T^*N$ at which we associate the following forms defined in natural coordinate (q^0, q^i, p^i):

$$\begin{aligned}\theta_h &= p_i dq^i + p_0 dq^0,\\ \Omega_h &= -dp_i \wedge dq^i - dp_0 \wedge dq^0.\end{aligned} \tag{25}$$

By analogy with (10) and (12) the Hamiltonian function for the mechanic is generally the following non-autonomous Hamiltonian:

$$h(t, q^i(t), p_i(t)) = (FL^{-1})^* L - p_i (FL^{-1})^* \dot{q}^i, \tag{26}$$

where

$$FL^* t = t, \quad FL^* q^i = q^i, \quad FL^* p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{27}$$

From (22), (25) and (26), we find naturally

$$p_0 = \frac{h}{c}. \tag{28}$$

In Eq. (22), we identify the terms

$$\begin{array}{ccc} \dfrac{\partial L}{\partial \dot{q}^i} dq^i = \Theta_L & \xrightarrow{FL} & p_i dq^i = \Theta_{\text{Liouville one-form}} \\ -d\downarrow & & -d\downarrow \\ \omega_L = -d\left(\dfrac{\partial L}{\partial \dot{q}^i}\right)\wedge dq^i & \xrightarrow{FL} & \omega = dq^i \wedge dp_i = \omega_{\text{canonical symplectic two-form}}. \end{array} \tag{29}$$

By substituting (27) in (25), we obtain (22):

$$FL^*\theta_h = \theta_L \quad \text{and} \quad FL^*\Omega_h = \Omega_L.$$

The corresponding multivector field equivalent to (23) which satisfies the Hamilton–Jacobi principal (14) is:

$$\overline{X}_h = \frac{\partial}{\partial q^0} + \overline{\dot{q}}^i \frac{\partial}{\partial q^i} + \overline{\dot{p}}^i \frac{\partial}{\partial p^i}. \tag{30}$$

The HDW equations obtained can be identified by the ODE equations for the curves in relativistic mechanics:

$$\begin{aligned} \frac{dq^i \circ \Psi}{dq^0} &= \overline{\dot{q}}^i = \left.\frac{\partial p_0}{\partial p_i}\right|_\Psi, \\ \frac{dp_i \circ \Psi}{dq^0} &= \overline{\dot{p}}_i = -\left.\frac{\partial p_0}{\partial q^i}\right|_\Psi. \end{aligned} \tag{31}$$

We put $Z \equiv (q^i, p_i), \nabla_Z \equiv (\partial_{q^i}, \partial_{p_i}); i = \overline{1,3}$, the system ODE of the HDW can be written by:

$$\begin{pmatrix} \dfrac{\partial p_0}{\partial q^i} \\ \dfrac{\partial p_0}{\partial p_i} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J_t} \begin{pmatrix} \partial_0 q^i = \overline{\dot{q}}^i \\ \partial_0 p_i = \overline{\dot{p}}_i \end{pmatrix} \quad \text{for } i \text{ fixed and } \partial_0 = \frac{\partial}{\partial q^0}. \tag{32}$$

The system (31) can be contracted naturally in:

$$\nabla_Z h_z = \underbrace{J_0 \partial_0}_{J_\partial} Z = J_0 \dot{Z}. \tag{33}$$

If we explicit the index $i = \overline{1,3}$ the system ODE (32) becomes:

$$\begin{pmatrix} \dfrac{\partial p_0}{\partial q_1} \\ \dfrac{\partial p_0}{\partial p_1} \\ \dfrac{\partial p_0}{\partial q_2} \\ \dfrac{\partial p_0}{\partial p_2} \\ \dfrac{\partial p_0}{\partial q_3} \\ \dfrac{\partial p_0}{\partial p_3} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{J_0} \begin{pmatrix} \overline{\dot{q}}_1 \\ \overline{\dot{p}}_1 \\ \overline{\dot{q}}_2 \\ \overline{\dot{p}}_2 \\ \overline{\dot{q}}_3 \\ \overline{\dot{p}}_3 \end{pmatrix}, \tag{34}$$

where J_0 is 6×6 skew-symmetric constant matrix and J_∂ depends only on the partial differential ∂_0 associated to the base space coordinates (q^0).

4. Relation between Multisymplectic Geometry and the k-Structure

4.1. *The multisymplectic geometry and k-cosymplectic structure*

We are going to concentrate ourselves on fields' theory; the result is the same for the mechanic theory, so: let the fiber bundle cited above be $E = R^4 \times N$. On the associated first-jet bundle $j^1\pi = R^4 \times T_4^1 N$, we have the local coordinate $(x_\mu, y^A, v_\mu^A); \mu = \overline{0,3}, A = \overline{1,d}$. Let the Lagrangian function $L \in C^\infty(J^1\pi)$ (i.e. $L(x_v, y^A, v_v^A)$) and $\chi_L^4(J^1\pi = R^4 \times T_4^1 N)$ be the set of the holonomic k-vector fields on $J^1\pi$ (see [9]), such that:

$$X_v = \frac{\partial}{\partial x^v} + F_v^A \frac{\partial}{\partial y^A} + G_{v\rho}^A \frac{\partial}{\partial v_\rho^A}, \tag{35}$$

where $F_v^A = v_v^A$ and $G_{v\rho}^A = \frac{\partial^2 y^A}{\partial x^v \partial x^\rho}$. These multivector fields, in four-cosymplectic Lagrangian formalism, are solutions for the following equation:

$$i_{X_v} \omega_L^v = -dE_L + \frac{\partial L}{\partial x^v} dx^v, \tag{36}$$

where

$$\omega_L^v = -d\left(\frac{\partial L}{\partial v_v^A} dy^A\right), \tag{37}$$

$$\Theta_L^v = \frac{\partial L}{\partial v_v^A} dy^A, \qquad E_L = -v_v^A \frac{\partial L}{\partial v_v^A} + L. \tag{38}$$

By putting (37) and (38) in (36), we obtain:

$$i_{X_v}\left[-d\left(\frac{\partial L}{\partial v_v^A} dy^A\right)\right] = d\left(v_v^A \frac{\partial L}{\partial v_v^A} - L\right) + \frac{\partial L}{\partial x^v} dx^v. \tag{39}$$

Because the base space is flat, we can use the following relations:

$$\begin{aligned} d[\alpha \wedge \beta] &= d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta, \\ i_X(\alpha \wedge \beta) &= i_X\alpha \wedge \beta + (-1)^r \alpha \wedge i_X\beta, \\ r &= d^\circ\alpha = \text{degrees of } \alpha. \end{aligned} \tag{40}$$

And "d" the total differential is defined on $J^1\pi$ by:

$$d = \frac{\partial}{\partial x^v} dx^v + \frac{\partial}{\partial y^A} dy^A + \frac{\partial}{\partial v_v^A} dv_v^A. \tag{41}$$

By multiplying (39) by the volume element $\eta = dx^k = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, we obtain:

$$\left\{ -i_{X^v} d\Theta_L^v - d\left(v_v^A \frac{\partial L}{\partial v_v^A} - L \right) - \frac{\partial L}{\partial x^v} dx^v \right\} \wedge d^k x = 0. \tag{42}$$

By using (40), the following term gives:

$$\begin{aligned} -i_{X^v} d\Theta_L^v \wedge dx^k &= -i_{X^v}(d\Theta_L^v \wedge dx^k) + (-1)^2 d\Theta_L^v \wedge i_{X^v} dx^k \\ &= -i_{X^v}[d(\Theta_L^v \wedge dx^k) - (-1)^1 \Theta_L^v \wedge d(dx^k)] \\ &\quad + d(\Theta_L^v \wedge i_{X^v} dx^k) - (-1)^1 \Theta_L^v \wedge d(i_{X^v} dx^k). \end{aligned} \tag{43}$$

By contracting (35) by η, we have:

$$i_{X^v} dx^k = dx_v^{k-1}. \tag{44}$$

Finally (43) gives:

$$-i_{X_v} d\Theta_L^v \wedge dx^k = -i_{X_v} d(\Theta_L^v \wedge dx^k) + d(\Theta_L^v \wedge dx_v^{k-1}). \tag{45}$$

The term

$$\frac{\partial L}{\partial x^v} dx^v \wedge dx^k = 0 \quad \forall \nu = \overline{0,3}. \tag{46}$$

Inserting (45) and (46) in (42), we obtain:

$$d\left[\Theta_L^v \wedge dx_v^{k-1} - \left(v_v^A \frac{\partial L}{\partial v_v^A} - L \right) \wedge dx^k \right] = i_{X^v} d(\Theta_L^v \wedge dx^k). \tag{47}$$

We identify the term in (47)

$$\Theta_L^v \wedge dx_v^{k-1} - \left(v_v^A \frac{\partial L}{\partial v_v^A} - L \right) \wedge dx^k = \theta_{\text{multisymplectic}}, \tag{48}$$

$$\Omega_{L\,\text{multisymplectic}} = -d\theta_{L\,\text{multisymplectic}} = -i_{X^v} d(\Theta_L^v \wedge dx^k). \tag{49}$$

4.2. *The multisymplectic geometry and k-symplectic structure*

By analogy with the work done for the k-cosymplectic structure, but in this case $E = N$, the Lagrangian function $L \in C^\infty(T_4^1 N)$ and $\chi_L^4(T_4^1 N)$ be the set of k-vector fields in $T_4^1 N$ (see [9]), such that:

$$\begin{aligned} X_v &= F_v^A \frac{\partial}{\partial y^A} + G_{v\rho}^A \frac{\partial}{\partial v_\rho^A}, \\ F_v^A &= v_v^A, \quad G_{v\rho}^A = \frac{\partial^2 y^A}{\partial x^v \partial x^\rho}. \end{aligned} \tag{50}$$

The relation (36) becomes:

$$i_{X_v}\omega_L^v = -dE_L, \tag{51}$$

where the term $\frac{\partial L}{\partial x^v} = 0$, $\forall\, v = \overline{0,3}$ and the expressions (37) and (38) of ω_L^v and E_L, Θ_L^v respectively seen in four-cosymplectic differ for four-symplectic in the expression of the Lagrangian (i.e. L does not depend on x^v) and "d" the total differential, in this case, is given by:

$$d = \frac{\partial}{\partial y^A}dy^A + \frac{\partial}{\partial v_v^A}dv_v^A. \tag{52}$$

Given the expression of k-vector field X_v, we have $i_{X^v}dx^k = 0$. The same calculus can be done for k-symplectic as it was done for four-cosymplectic, but in this case the term $(\Theta_L^v \wedge dx_v^{k-1})$ disappears automatically from (48) which it becomes

$$-\left(v_v^A\frac{\partial L}{\partial v_v^A} - L\right)\wedge dx^k \neq \theta_{\text{multisymplectic}}. \tag{53}$$

We can make the following remark that the contribution of the term $\frac{\partial L}{\partial x^v}$ vanishes in both cases if it is k-symplectic or k-cosymplectic. So, to study the dynamic of a k-symplectic physical system, we can recalculate demonstration for the relationship between k-symplectic and multisymplectic treating the dynamic in the first-jet bundle $j^1\pi = R^k \times T_k^1 N$ as if it was k-cosymplectic with $\frac{\partial L}{\partial x^v} = 0$.

In conclusion, we deduce that the multisymplectic geometry is k-cosymplectic structure such that the fibers of the first-jet bundle $J^1\pi$ for the Lagrangian formalism are constructed based on the two-symplectic form $\omega_L^v = -d(\frac{\partial L}{\partial v_v^A}dy^A)$ defined on each point of the fiber. So, ω_L^v forms the structure of the multisymplectic geometry.

We remark also that, in physics, if the theory is explicit so the Lagrangian depends on the local coordinates $(x_v, y^A = \phi^A(x_v), v_v^A = \partial_v\phi^A(x_v))$ (i.e. $L(x_v, y^A, v_v^A)$ and $\frac{\partial L}{\partial x^v} \neq 0$), the structure of the geometry is k-cosymplectic and the dynamic of fields is studied on a first-jet bundle $J^1\pi = R^4 \times T_4^1 N$. But if the Lagrangian depends only on the following coordinates (y^A, v_v^A) and $\frac{\partial L}{\partial x^v} = 0$, $\forall\, v = \overline{0,3}$, the theory is said to be implicit and the structure is k-symplectic and can be also constructed on the fiber bundle $J^1\pi = R^4 \times T_4^1 N$ above the database R^4:

$$FL : R^4 \times T_4^1 N \to R^4 \times (T_4^1)^* N.$$

By using the Legendre map

$$(x_v, y^A, v_v^A) \to \left(x_v, y^A, p_v^A = \frac{\partial L}{\partial v_v^A}\right), \tag{54}$$

without any demonstration, we can use the same method done for the Lagrangian formalism and by using FL; the same results will be deduced for the Hamiltonian formalism. In this case, the k-cosymplectic structure of the geometry is based on the symplectic two-form

$$\omega_v = dy_A \wedge dp_v^A = FL\, \omega_L^v.$$

We can use the same method of demonstration and without doing any calculus using the fiber bundle cited above in Sec. 3 for the relativistic mechanic, we will find a result similar to Eq. (49) just by making the following change of variables:

$$(x_v, y^A, v_v^A) \quad \text{and} \quad k = 4 \to (ct, q^i, \dot{q}^i) \quad \text{and} \quad k = 1. \tag{55}$$

We can construct the multisymplectic geometry for the relativistic mechanics based onto the canonical symplectic two-form (29), Eq. (36) becomes in this case:

$$i_{\overline{X}}\omega_L = -dE_L + \frac{\partial L}{\partial t} dt. \tag{56}$$

Unless, in this case, we can make the following remark that the relativistic mechanic is implicit (i.e. $\frac{\partial L}{\partial t} = 0$ and $L(q^i, \dot{q}^i)$), the theory is said to be autonomous. If the theory of relativistic mechanics is explicit (i.e. $\frac{\partial L}{\partial t} \neq 0$ and $L(t, q^i, \dot{q}^i)$), it is called non-autonomous.

5. Dynamic of Physical Systems

By using the multisymplectic geometry studied in sections cited above, and having regard to the relationship between the k-cosymplectic (k-symplectic respectively) and multisymplectic geometry in Sec. 4. Here, we are going to develop the model for studying separately the dynamic of a relativistic charged particle immerged in a weak field: the electromagnetic field $A_\mu \equiv (\phi, A_i)$ and in the absence of the gravitational field (a strong field), then the dynamic of a classical free electromagnetic field.

5.1. *Dynamic of a relativistic charged particle*

A geometric formulation of the Lagrangian of relativistic mechanics in terms of jets bundle was already treated in [38–40], but in this subsection, we are going to treat the Hamiltonian of relativistic mechanics in the geometrical model cited above in Sec. 3.

The Lagrangian function of a relativistic particle immerged in an electromagnetic field

$$\begin{aligned} L(t, q^i, \dot{q}^i) &= m_0 c \sqrt{\dot{q}_\mu \dot{q}^\mu} + e \dot{q}_\mu A^\mu(t, q^i), \\ L(t, q^i, \dot{q}^i) &= m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + e\phi(t, q^i) - e\vec{v} \cdot \vec{A}(t, q^i), \end{aligned} \tag{57}$$

where m_0 is the rest mass of the particle. Where the four-vector $\dot{q}_\mu$ is given by:

$$\dot{q}_\mu = \frac{dq_\mu}{dt} = \begin{cases} \dot{q}_0 = c, \\ (\dot{q}_i)_{i=1,2,3} = \vec{v}. \end{cases} \tag{58}$$

By using the metric $\eta^{\mu\nu}$ defined in Minkowski space in Sec. 3, the four-vector of the electromagnetic field is

$$A^\mu = \begin{cases} A^0 = A_0 = \dfrac{\phi}{c}, \\ (A^i)_{i=1,2,3} = -(A_i)_{i=1,2,3} = -\vec{A}. \end{cases} \tag{59}$$

The four-momentum vector

$$p^\mu = \frac{\partial L}{\partial \dot{q}_\mu} = m_0 c \frac{\dot{q}^\mu}{\sqrt{\dot{q}_v \dot{q}^v}} + eA^\mu = \begin{cases} p^0 = \dfrac{m_0 c}{\sqrt{1-\frac{v^2}{c^2}}} + e\dfrac{\phi}{c} = \dfrac{h}{c}, & \text{(60a)} \\ p^i = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \dot{q}^{\,i} + eA^i. & \text{(60b)} \end{cases}$$

We do the following change of coordinate

$$\begin{cases} P^0 = p^0 - eA^0 = mc, & \text{(61a)} \\ P^i = p^i - eA^i = m\dot{q}^{\,i}, & \text{(61b)} \end{cases}$$

where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ and P^0, p^0 are respectively the mass and the energies (the Hamiltonians in different system of coordinates) of the particle moving at the speed $\vec{v}$.

The Hamiltonian function for a relativistic particle is given by:

$$p^\mu \dot{q}_\mu - L = (p^\mu - eA^\mu)(p_\mu - eA_\mu) - m_0 c^2 = 0 \quad \text{and} \quad p^\mu = \frac{\partial L}{\partial \dot{q}_\mu}. \tag{62}$$

By substituting (57) and (61) in (62), we obtain:

$$P^0 = p^0 - eA^0 = \frac{E}{c} - e\frac{\phi}{c} = \sqrt{m_0 c^2 - (p^i - eA^i)(p_i - eA_i)}. \tag{63}$$

Having regard to the relationship (61a) between $\dot{q}^{\,i}$ and p^i, more of this, seeing the relation (63), only the components $\{\dot{q}^i\}_{i=\overline{1,3}}$ and $\{p^i\}_{i=\overline{1,3}}$ respectively are independents, so, the multisymplectic model cited in Sec. 3.2 and the result found in Sec. 4.2 are valid for the study of the dynamics in the jet bundle $J^1\pi = R \times TR^3$

generated by $(ct, q^i, \dot{q}^i)$:

$$\frac{\partial^2 L}{\partial \dot{q}^\mu \partial \dot{q}^v} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\left[\eta^{\mu v} + \frac{\dot{q}_\mu \dot{q}_v}{c^2 - v^2}\right] \neq 0, \quad \forall\, \mu = v = \overline{0,3}.$$

The Lagrangian is hyper-regular. The Lagrangian formalism in the first-jet bundle $J^1\pi$ is equivalent to a Hamiltonian one treated on the first-jet bundle $J^1\pi^*$ generated by the following local coordinate system (ct, q^i, p^i) as in Sec. 3.2. The Hamilton–Jacobi principal (14) is verified for:

$$\begin{aligned} \Omega_h &= -dp_i \wedge dq^i - dp_0 \wedge dq^0, \\ \overline{X}_h &= \frac{\partial}{\partial q^0} + \overline{\dot{q}}^i \frac{\partial}{\partial q^i} + \overline{\dot{p}}^i \frac{\partial}{\partial p^i}. \end{aligned} \tag{64}$$

Because the laws of physics must be invariant of a system of coordinates to another, in this case, the dynamic of the particle can be studied on the first-jet bundle $J^1\pi^* = R \times T^*N$ generated by the following coordinate system (ct, q^i, P^i). The Hamilton–Jacobi principal (14) becomes:

$$\begin{aligned} \tilde{\Omega}_h &= -dP_i \wedge dq^i - dP_0 \wedge dq^0 = -dP_\mu \wedge dq^\mu, \\ \tilde{X}_h &= \frac{\partial}{\partial q^0} + \overline{\dot{q}}^i \frac{\partial}{\partial q^i} + \overline{\dot{P}}^i \frac{\partial}{\partial P^i}. \end{aligned} \tag{65}$$

By using the change of coordinates (61), we have

$$\tilde{\Omega}_h = -dP_\mu \wedge dq^\mu = -dp_\mu \wedge dq^\mu + edA_\mu \wedge dq^\mu. \tag{66}$$

By using $dq^\mu \wedge dq^v = -dq^v \wedge dq^\mu,\ \forall\, \mu, v = \overline{0,3}$,

$$\begin{aligned} dA_\mu \wedge dq^\mu &= dA_0 \wedge dq^0 + dA_i \wedge dq^i \\ &= \frac{\partial A_0}{\partial q^i} dq^i \wedge dq^0 + \frac{\partial A_i}{\partial q^j} dq^j \wedge dq^i + \frac{\partial A_i}{\partial q^0} dq^0 \wedge dq^i \\ &= \left(\frac{\partial A_i}{\partial q^0} - \frac{\partial A_0}{\partial q^i}\right) dq^0 \wedge dq^i + \frac{1}{2}\left(\frac{\partial A_i}{\partial q^j} - \frac{\partial A_j}{\partial q^i}\right) dq^j \wedge dq^i \\ &= \left[F_{0i}\, dq^0 + \frac{1}{2} F_{ji}\, dq^j\right] \wedge dq^i \\ &= \frac{1}{2} F_{\mu\nu}\, dq^\mu \wedge dq^v = \frac{1}{2} F_{\nu\mu}\, dq^v \wedge dq^\mu. \end{aligned} \tag{67}$$

Inserting (67) in (66)

$$\begin{aligned} \tilde{\Omega}_h &= -dP_\mu \wedge dq^\mu = -\left(dp_\mu + \frac{e}{2} F_{\mu\nu}\, dq^v\right) \wedge dq^\mu \\ &= -dp_0 \wedge dq^0 - dp_i \wedge dq^i - \frac{e}{2} F_{ij}\, dq^j \wedge dq^i - eF_{0i}\, dq^i \wedge dq^0. \end{aligned} \tag{68}$$

By using the contraction (38):

$$\left(\frac{\partial}{\partial q^\rho}\right) dq^\mu = \delta^\mu_\rho \quad \text{and} \quad \left(\frac{\partial}{\partial q^\rho}\right)(dq^\mu \wedge dq^v)dq^\mu = \delta^\mu_\rho dq^v - \delta^v_\rho dq^\mu. \tag{69}$$

The Hamilton–Jacobi principal (14) gives:

$$\begin{aligned} i(\tilde{X}_h)\tilde{\Omega}_h = {} & \frac{\partial p_0}{\partial q^i}dq^i + \frac{\partial p_0}{\partial p^i}dp^i - e\,F_{i0}dq^i - \bar{\dot{q}}^i \frac{\partial p_0}{\partial q^j}\delta^j_i dq^0 + \bar{\dot{q}}^j \delta^i_j dp_i \\ & - \frac{e}{2}F_{ij}\left[\bar{\dot{q}}^\rho \delta^j_\rho dq^i - \bar{\dot{q}}^\rho \delta^i_\rho dq^j\right] + e\,F_{i0}\bar{\dot{q}}^j \delta^i_j dq^0 \\ & - \bar{\dot{P}}^j \frac{\partial p_0}{\partial p^i}\delta^j_i dq^0 - \bar{\dot{P}}_j \delta^j_i dq^i = 0. \end{aligned} \tag{70}$$

Knowing that $\frac{\partial}{\partial P^i}(dp^j) = \delta^j_i$ and $P^i = p^i - eA$. By collecting the terms in (70):

$$\begin{aligned} i(\tilde{X}_h)\tilde{\Omega}_h = {} & \left(e\,F_{i0}\bar{\dot{q}}^i - \bar{\dot{P}}^i \frac{\partial p_0}{\partial p^i} - \bar{\dot{P}}_i \frac{dq^i}{dq^0} - \bar{\dot{q}}^i \frac{\partial p_0}{\partial q^i}\right) dq^0 \\ & + \left(\frac{\partial p_0}{\partial q^i} - e\,F_{i0} - e\bar{\dot{q}}^j F_{ij}\right) dq^i + \left(\frac{\partial p_0}{\partial p^i} + \bar{\dot{q}}_i\right) dp^i = 0. \end{aligned} \tag{71}$$

By geometric construction that was made and without going through the derivation of the Hamiltonian (63), if we identify term-by-term in equality (71), the system of the HDW equations (31) which describes the dynamic of a particle plunged in electromagnetic field gives:

$$\begin{cases} \dfrac{\partial p_0}{\partial p^i} = -\bar{\dot{q}}_i = \bar{\dot{q}}^i, & \text{(72a)} \\ -\dfrac{\partial p_0}{\partial q^i} = e\,F_{0i} - e\bar{\dot{q}}^j F_{ij} = e(F_{0i} + \bar{\dot{q}}_j F_{ij}) = \bar{\dot{p}}_i. & \text{(72b)} \end{cases}$$

Inserting Eqs. (72a) and (72b) in the first bracket of the relation (71),

$$e\,F_{i0}\bar{\dot{q}}^i - \bar{\dot{P}}^i \frac{\partial p_0}{\partial p^i} - \bar{\dot{P}}_i \frac{dq^i}{dq^0} - \bar{\dot{q}}^i \frac{\partial p_0}{\partial q^i} = -e\bar{\dot{q}}^i \bar{\dot{q}}^j F_{ij} = 0.$$

By multiplying the system (72) by "c",

$$\begin{cases} \dfrac{\partial h}{\partial p^i} = \dot{q}^i, & \text{(73a)} \\ -\dfrac{\partial h}{\partial q^i} = e(\vec{E} + \vec{\dot{q}} \wedge \vec{B})_i = \vec{F}_{i\,\text{Lorentz force}} = \dot{p}_i. & \text{(73b)} \end{cases}$$

In particular $\bar{\dot{P}}_i = -\frac{\partial P_0}{\partial q^i} = e\frac{\partial A_i}{\partial q^0} - e\bar{\dot{q}}^j F_{ij}$.

We conclude that such a geometrical construction lets one to identify the Hamilton–Carton two-form $\tilde{\Omega}_h$ (65) to the charged two-symplectic form already used in the work [41]. This geometrical model is invariant by boosts when moving from one fiber to another horizontally on Minkowski space which coincides, in this model, with the configuration bundle E. The relation (68) shows how changes $\tilde{\Omega}_h$ from the coordinate system (ct, q^i, P^i) to the other one (ct, q^i, p^i) which is the result already put in [41] where $\tilde{\Omega}_h$(65) was defined on the base space M taken as Minkowski space.

We can also do the following remark that the contribution of the term $\frac{\partial p^0}{\partial q^0}$ disappears naturally in the expression of $\tilde{\Omega}_h$ (71). This is explained by the fact that the expression (63) of p^0 is time implicit.

The classical limit which is the development to the first order of the theory leaves invariant the model built where just tender the terms $\frac{v}{c} \to 0$ and

$$\frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\alpha}} = \left(1 - \frac{v^2}{c^2}\right)^{-\alpha} = 1 + \alpha\frac{v^2}{c^2}.$$

The classical limit of the term

$$\left[1 - \frac{1}{c^2\left(1 - \frac{v^2}{c^2}\right)}\dot{q}^i\dot{q}_i\right]^{1/2} = 1 + \frac{v^2}{2c^2} = 1 + \frac{(p_i - eA_i)^2}{2c^2}. \tag{74}$$

Inserting (74) in (61b) and (63), the classical limit of the Hamiltonian of the particle

$$h = m_0c^2 + \frac{m_0}{2}(p_i - eA_i)^2 + e\phi, \tag{75}$$

if we take the origin of energies $h_0 = m_0c^2$. Inserting (60b) in (73b), we retrieve the Newton's law

$$m_0\ddot{q}^i = -e\partial_i\phi + e(\vec{\dot{q}} \wedge \vec{B})_i. \tag{76}$$

In this case, the theory is invariant by Galilean transformation which is the classical limit of boosts.

By using the result found in Sec. 4.2, the model built in Sec. 3.2 remains valid for the study of the dynamics of free particle (i.e. $A_\mu = 0$) in the jet bundle $J^1\pi^* = R \times T^*R^3$ generated by (ct, q^i, p^i) and the last remark that we can do is that the multisymplectic Hamiltonian formalism for relativistic mechanics is the familiar homogeneous Hamiltonian formalism of non-relativistic mechanics.

5.2. *Dynamic of the free electromagnetic field*

Recall the expression of the Lagrangian of a free electromagnetic field

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu v},$$

where $F_{\mu\nu} = \partial_\mu A_v - \partial_v A_\mu$, $A_\mu(x^\rho)$ is the electromagnetic field and $v_{\mu\nu} = \partial_\mu A_v$ is the velocity of the field $A_v, \partial_\mu = \frac{\partial}{\partial x^\mu}$. Having regarded to the expression of the

Lagrangian $L(\partial_\mu A_v)$ and x^μ is implicit, the theory is four-symplectic. By using the result obtained in Sec. 4.2, we can study the dynamic of the field, by using the multisymplectic geometry studied in Sec. 2.2, on $J^1\pi^* = R^4 \times (T_4^1)^* N$ where N is the space of the field A_μ where group of symmetry is $U(1)$.

$$\begin{aligned} L &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}(\partial_\mu A_v - \partial_v A_\mu)(\partial^\mu A^v - \partial^v A^\mu) \\ &= -\frac{1}{2}[(\partial_\mu A_v)(\partial^\mu A^v) - (\partial^v A^\mu)(\partial_\mu A_v)] = -\frac{1}{2}(\partial_\mu A_v)F^{\mu\nu}, \qquad (77) \\ p_{\mu\nu} &= \frac{\partial L}{\partial v^{\mu\nu}} = -F_{\mu\nu} = F_{\nu\mu}, \quad \frac{\partial L}{\partial x^\mu} = 0. \end{aligned}$$

The Hamiltonian of the field

$$\begin{aligned} h &= L - v_{\mu\nu}\frac{\partial L}{\partial v_{\mu\nu}} = -\frac{1}{2}(\partial_\mu A_v)F^{\mu\nu} - (\partial_\mu A_v)(-F^{\mu\nu}) = \frac{1}{2}(\partial_\mu A_v)F^{\mu\nu} \\ &= \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{4}p^{\mu\nu}p_{\mu\nu} = h(p_{\mu\nu}) \qquad (78) \end{aligned}$$

so,

$$dh = \frac{\partial h}{\partial p^{\mu\nu}}dp^{\mu\nu} = F_{\mu\nu}dF^{\mu\nu}.$$

By using (13) and (15), we obtain

$$\begin{aligned} \Omega_h &= -dp^{\mu\nu} \wedge dA_v \wedge d^3x_\mu - dh \wedge d^4x \\ &= dF_{\mu\nu} \wedge dA^v \wedge d^3x^\mu - F^{\mu\nu}dF_{\mu\nu} \wedge d^4x \\ &= dF^{\mu\nu} \wedge \frac{\partial A_v}{\partial x^\rho}dx^\rho \wedge d^3x_\mu - F_{\mu\nu}dF^{\mu\nu} \wedge d^4x \\ &= \partial_\mu A_v dF^{\mu\nu} \wedge d^4x - F_{\mu\nu}dF^{\mu\nu} \wedge d^4x \\ &= \partial_v A_\mu dF^{\mu\nu} \wedge d^4x = -\frac{1}{2}F_{\mu\nu}dF^{\mu\nu} \wedge d^4x \qquad (79) \end{aligned}$$

and

$$X_h = \frac{\partial}{\partial x^\rho} + S_\rho{}^\sigma \frac{\partial}{\partial A^\sigma} - G^{\sigma\gamma}{}_\rho \frac{\partial}{\partial F^{\sigma\gamma}}.$$

Knowing that $dF^{\mu\nu}$ is one-form on $J^1\pi^*$ and by using (40), we have

$$i\left(\frac{\partial}{\partial x^\rho}\right)\Omega_h = -\frac{1}{2}[\partial_\rho F_{\mu\nu} \wedge dF^{\mu\nu} \wedge dx^4 + F_{\mu\nu}(\partial_\gamma F^{\mu\nu})\delta_\rho^\gamma d^4x + F^{\mu\nu}dF_{\mu\nu} \wedge d^3x_\rho].$$

The Hamilton–Jacobi principal (14) gives:

$$\begin{aligned} i(X_h)\Omega_h &= -\frac{1}{2}[\partial_\rho F_{\mu\nu} \wedge dF^{\mu\nu} \wedge dx^4 + F_{\mu\nu}(\partial_\gamma F^{\mu\nu})\delta_\rho^\gamma d^4x + F^{\mu\nu} dF_{\mu\nu} \wedge d^3x_\rho] \\ &\quad - \frac{1}{2}(-G^{\sigma\gamma}{}_\rho)\delta_\mu^\sigma \delta_v^\gamma dF^{\mu\nu} \wedge dx^4 \\ &= (\partial_\rho F^{\mu\nu} - G^{\mu\nu}{}_\rho) dF_{\mu\nu} \wedge dx^4 + 2F_{\mu\nu}\partial_\rho F^{\mu\nu} d^4x = 0. \end{aligned} \tag{80}$$

By identification the terms in (80), we obtain

$$\partial_\rho F^{\mu v} - G^{\mu v}{}_\rho = 0 \Rightarrow G^{\mu v}{}_\rho = \partial_\rho F^{\mu v}, \quad \text{in particular for } \rho = \mu, \quad \underbrace{\partial_\mu F^{\mu v} = 0 = \frac{\partial h}{\partial A^v}}$$

$$\Downarrow$$

$$\underbrace{\Box A^v - \underbrace{\partial^v \partial_\mu A^\mu}_{\text{Lorentz gauge}=0} = 0}$$

$$\Downarrow$$

$$\Box A^v = 0 \begin{pmatrix} \text{propagation equation for} \\ \text{the free electromagnetic} \\ \text{field } A^v \end{pmatrix},$$

$$2F_{\mu v}(\partial_\rho F^{\mu v}) d^4x = \partial_\rho (F_{\mu v} F^{\mu v}) d^4x = 0 \Rightarrow h = \text{constant}. \tag{81}$$

Because the k-vector-field X_h is holonomic (SOPDE and integrable), $S_\rho{}^\sigma = \partial_\rho A^\sigma$.

In conclusion of this Sec. 5, we deduce that the multisymplectic geometry is the favorable geometry for describing both the relativistic dynamics for gauge theories (boson fields) (i.e. the electromagnetic field) which dynamic obeys at Maxwell equations, e.g. Sec. 5.2 and the mechanic which obeys at Newton equations in the classical limit, e.g. Sec. 5.1 respectively.

6. Conclusion

After this study, we deduce that the extension of the multisymplectic geometry to the relativistic mechanics operates successfully. Furthermore, we have proved that in the absence of the gravitational field, both the propagation of a free field (the movement of a free particle or a particle immerged in a weak field as the electromagnetic field) on the space respectively lead to a construction of a multisymplectic geometry on the first-jet bundle whose base space is treated globally. Finally, we have found a direct relationship between the k-cosymplectic structure and the multisymplectic geometry.

Acknowledgment

The authors would like to thank Professor Carl M. Bender, Professor of Physics at Washington University, for his valuable help in providing us with a list of topics related to our field of interest.

References

[1] A. Awane, k-symplectic structures, *J. Math. Phys.* **32**(12) (1992) 4046–4052.

[2] A. Awane, G-spaces k-symplectic homogènes, *J. Geom. Phys.* **13** (1994) 139–157.

[3] A. Awane and M. Goze, *Pfaffian Systems, k-symplectic Systems* (Kluwer Academic Publishers, Dordrecht, 2000).

[4] C. Günther, The polysymplectic Hamiltonian formalism in the field theory and the calculus of variations I: The local case, *J. Differential Geom.* **25** (1987) 23–53.

[5] M. De Leon, E. Merino, J. A. Oubina, P. Rodrigues and M. Salgado, Hamiltonian systems on k-cosymplectic manifolds, *J. Math. Phys.* **39**(2) (1998) 876–893.

[6] M. De Leon, E. Merino and M. Salgado, k-cosymplectic manifolds and Lagrangian field theories, *J. Math. Phys.* **42**(5) (2001) 2092–2104.

[7] L. K. Norris, Generalized symplectic geometry on the frame bundle of a manifold, *Proc. Sympos. Pure Math.* **54**(2) (1993) 435–465.

[8] L. K. Norris, n-symplectic algebra of observables in covariant Lagrangian theory, *J. Math. Phys.* **42**(10) (2001) 4827–4845.

[9] M. De Leon, M. McLean, L. K. Norris, A. Rey-Roca and M. Salgado, Geometric structures in field theory, preprint (2002), arXiv: math-ph/0208036.

[10] M. Castrillon-Lopez and J. E. Marsden, Some remarks on Lagrangian and Poisson reduction for field theories, *J. Geom. Phys.* **48**(1) (2003) 52–83.

[11] M. J. Gotay, A multisymplectic framework for classical field theory and the calculus of variations I: Covariant Hamiltonian formalism, in *Mechanics, Analysis and Geometry*: 200 *Years after Lagrange*, ed. M. Francaviglia (Elsevier Science Publisher, 1991), pp. 203–235.

[12] M. J. Gotay, J. Isenberg and J. E. Marsden, Momentum maps and classical relativistic fields I: Covariant theory, preprint (1998), arXiv: physics/9801019.

[13] J. K. Lawson, A frame bundle generalization of multisymplectic geometries, *Rep. Math. Phys.* **45**(2) (2000) 183–205.

[14] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, *Commun. Math. Phys.* **30** (1973) 99–128.

[15] J. Kijowsky and W. Szczyrba, Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory, in *Géométrie Symplectique et Physique Mathématique, Colloques Internationaux du Centre, National de la Recherche Scientifique*, Vol. 237 (editions du CNRS, 1975), pp. 347–378.

[16] J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics, Vol. 170 (Springer-Verlag, Berlin, 1979).

[17] V. Aldaya and J. A. De Azcarraga, Geometric formulation of classical mechanics and field theory, *Riv. Nuovo Cimento* **3**(10) (1980) 1–66.

[18] E. Binz, J. Sniatycki and H. Fisher, *The Geometry of Classical Fields* (North-Holland, Amsterdam, 1988).

[19] A. Echeverria-Enriquez, M. C. Muñoz-Lecanda and N. Roman-Roy, Geometry of Lagrangian first-order classical field theories, *Fortschr. Phys.* **44** (1996) 235–280.

[20] A. Echverria-Enriquez, M. C. Muñoz-Lecanda and N. Roman-Roy, Multivector fields and connections: Setting Lagrangian equations in field theories, *J. Math. Phys.* **39**(9) (1998) 4578–4603.

[21] G. Giachetta, L Mangiarotti and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific Publishing, Singapore, 1997).

[22] D. J. Saunders, The geometry of jet bundles, London Mathematical Society Lecture Notes Series, Vol. 142 (Cambridge, University Press, 1989).

[23] N. Roman-Roy, Multisymplectic Lagrangian and Hamiltonian formalisms of first-order classical field theories, preprint (2005), arXiv: math-ph/0506022.

[24] A. Echeverria-Enriquez, M. C. Muñoz-Lecanda and N. Roman-Roy, Geometry of multisymplectic Hamiltonian first-order field theories, *J. Math. Phys.* **41**(11) (2000) 7402–7444.

[25] A. Echeverria-Enriquez, M. De Leon, M. C. Muñoz-Lecanda and N. Roman-Roy, Hamiltonian systems in multisymplectic field theories, preprint (2005), arXiv: math-ph/0506003.

[26] A. Echeverria-Enriquez and M. C. Muñoz-Lecanda, Variational calculus in several variables: A Hamiltonian approach, *Ann. Inst. H. Poincaré* **56**(1) (1992) 27–47.

[27] F. Hélein and J. Kouneiher, Finite dimensional Hamiltonian formalism for gauge and quantum field theories, *J. Math. Phys.* **43**(5) (2002) 2306–2347.

[28] F. Hélein and J. Kouneiher, Covariant Hamiltonian formalism for the calculus of variations with several variables: Lepage–Dedecker versus De Donder–Weyl, *Adv. Theor. Math. Phys.* **8**(3) (2004) 565–601.

[29] J. E. Marsden, S. Pekarsky, S. Shkoller and M. West, Variational methods, multisymplectic geometry and continuum mechanics, *J. Geom. Phys.* **38**(3–4) (2001) 253–284.

[30] C. Paufler and H. Romer, Geometry of Hamiltonian n-vector fields in multisymplectic field theory, *J. Geom. Phys.* **44**(1) (2002) 52–69.

[31] A. Echeverria-Enriquez, M. C. Muñoz-Lecanda and N. Roman-Roy, Multivector fields and connections: Applications to field theories, *Rev. R. Acad. Scienc. Exact as Fis. Nat. Ser. A Mat.* **96**(1) (2002) 83–93.

[32] J. F. Cariñena, M. Crampin and L. A. Ibort, On the multisymplectic formalism for first order field theories, *Differential Geom. Appl.* **1** (1991) 345–374.

[33] M. De Leon, J. Marin-Salano and J. C. Marrero, A geometrical approach to classical field theories: A constraint algorithm for singular theories, in *Proc. New Developments in Differential Geometry*, eds. L. Tamassi and J. Szenthe (Kluwer Academic Press, 1996), pp. 291–312.

[34] G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory*: *Constraint Systems* (World Scientific, Singapore, 1995).

[35] J. B. Thomas, *Proc. Roy. Soc. London A* **462** (2006) 1531–1551.

[36] H. Loumi and M. Tahiri, *Rep. Math. Phys.* **33** (1993) 367–373.

[37] J. B. Thomas and R. Sebastian, Numerical methods for Hamiltonian PDEs, *J. Phys. A*: *Math. Gen.* **39** (2006) 5287–5320.

[38] G. Sardanashvily, Relativistic mechanics in a general setting, *Int. J. Geom. Math. Mod. Phys.* **7** (2010) 1307–1319.

[39] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Geometric Formulation of Classical and Quantum Mechanics* (World Scientific, Singapore, 2010).

[40] G. Sardanashvily, Lagrangian dynamics of submanifolds: Relativistic mechanics, *J. Geom. Mech.* **4** (2012) 99–110.

[41] L. K. Norris, On the affine connection-structure of the charged symplectic 2-form, *Int. J. Theor. Phys.* **30** (1991) 1127–1150.