People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Science and Technology, Mohamed Boudiaf, Oran

Faculty of Physics *Department of Physics*

"Physics of a material point"

Course and exercises with solutions

Handout

Intended for 1st year S.T and S.M students

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FOREWORD

This course handout is intended for first-year students of the Licence-Master-Doctorate (LMD) system, specialty: Materials Science (SM) and Science and Technology (ST). It follows the program taught in the departments of Physics within the Algerian Universities.

This course handout contains course reminders and solved exercises on the different chapters of the Physics 1 module "Mechanics of the point".

Tutorials were introduced to emphasize the basic concepts of classical material point mechanics. The exercises accompanying these course reminders have been chosen for the purpose of providing effective training to students in order to facilitate their understanding of the course and consolidate their knowledge.

This course handout is divided into five chapters:

Chapter - 1: *Dimensional Equations, Calculation of Uncertainties and Vectorial Calculation*
Chapter - 2: Kinematics
Chapter - 3: Relative Motion
Chapter - 4: Dynamics of the material point
Chapter - 5: Work and Energy

I would like to emphasize that this course handout is not a substitute for lectures and tutorials. The presence of the student and his interaction with his teacher is irreplaceable. This handout serves as a support and guide for the student in his learning and knowledge journey.

I wish all our students a good university experience and a path full of success.

Dr. Sid Ahmed Sfiat

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Chapter 1

Dimensional Equations, Uncertainties and Vector Calculus

1.1 Dimensional Equations:

Dimension is the physical quantity associated with a physical object regardless of the unit used for measuring the object. There are seven basic quantities of the international system chosen by physicists, from which all the quantities of physics can be formed.

Fundamental dimension	mass	Length	Time	Electrical intensity	Temperature	Quantity of matter	Light intensity
Unit (SI)	Kg	m	s	A	k	Mol	Cd
Symbol	M	L	T	I	Ө	N	J

A physical constant is a dimensioned quantity. A numeric constant is dimensionless.
Do not confuse dimension and unit. Indeed, a physical quantity has one and only one dimension but it can be expressed in several systems of different units.
Two physical quantities are homogeneous if they have the same dimension. One can only add or subtract quantities of the same dimension and expressed in the same system of units.
Dimensional analysis makes it possible to find the dimension and unit of a quantity if we know a simple equation linking this quantity to others of known dimension.
Here are some rules to follow in order to find the dimension and the unit of a physical quantity :

- Square brackets [...] are used to express the dimension of the object considered.
- The arguments of mathematical functions are dimensionless.
- If $A = B^n \times C^m$, alors $[A] = [B]^n \times [C]^m$.
- $[\frac{d^nA}{dx^n}] = \frac{[A]}{[x]^n}$ and $[\int Adx] = [A] . [X\}$
- The dimension of a quantity G is given according to the 7 fundamental dimensions :
  $[G] = \mathbf{M^x L^y T^z \Theta^a I^b N^c J^d}$ with real numbers as exponents.

The following table illustrates the different quantities formed on the basis of fundamental quantities :

Size	Relationship	dimension	Base unit/symbol
Length	L	**L**	m
Surface	$S = L^2$	$\mathbf{L^2}$	m^2
Volume	$V = L^3$	$\mathbf{L^3}$	m^3
Time	t	**T**	s
Speed	$\mathrm{v} = d/t$	$\mathbf{LT^{-1}}$	ms^{-1}
Acceleration	$a = \mathrm{v/t}$	$\mathbf{LT^{-2}}$	ms^{-2}
Frequency	$\nu = 1/T$	$\mathbf{T^{-1}}$	s^{-1} or Hertz (Hz)
mass	m	**M**	kg

Density	ρ = m/V	$\mathbf{ML^{-3}}$	kgm^{-3}
Force	F = ma	$\mathbf{MLT^{-2}}$	$kgms^{-2}$ or Newton (N)
Energy	$E = mc^2$	$\mathbf{ML^{2}T^{-2}}$	$kgm^{2}s^{-2}$ or Joule (J)
Power	P = E/t	$\mathbf{ML^{2}T^{-3}}$	$kgm^{2}s^{-3}$ or Watt (W)
Pressure	P = F/S	$\mathbf{ML^{-1}T^{-2}}$	$kgm^{-1}s^{-2}$ or Pascal(Pa)
Current Intensity	I	**I**	Ampere (A)
Electrical charge	Q = It	**IT**	$A \cdot s$ or Coulomb (C)
Voltage	U = P/I	$\mathbf{ML^{2}T^{-3}I^{-1}}$	$kgm^{2}s^{-3}A^{-1}$ or Volt(V)
Resistance	R = U/I	$\mathbf{ML^{2}T^{-3}I^{-2}}$	$kgm^{2}s^{-3}A^{-2}$ or ohm (Ω)
Capacity	C = Q/U	$\mathbf{M^{-1}L^{-2}T^{4}I^{2}}$	$kg^{-1}m^{-2}s^{4}A^{2}$ or farad (f)
Inductance	U = Ldi / dt	$\mathbf{ML^{2}T^{-2}I^{-2}}$	$kgm^{2}s^{-2}A^{-2}$ or henry (h

The following table shows the different multiples used with the units :

Multiple	10^{-15}	10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^{-1}
Prefix	femto	pico	nano	microphone	milli	centi	deci
Symbol	F	p	n	µ	m	c	d

Multiple	10^{1}	10^{2}	10^{3}	10^{6}	10^{9}	10^{12}	10^{15}
Prefix	deca	hecto	kilo	mega	giga	tera	peta
Symbol	Da	h	K	M	G	T	P

Exercises :

1. Find the pulsation ω of the simple pendulum knowing that it is a function of the gravitational acceleration g , the length of the wire l and the mass m of the material point attached to the wire.
   - $\omega = \alpha m^a l^b g^c$ where α is a dimensionless numeric constant.
     Then, $[\omega] = [m]^a[l]^b[g]^c = \mathbf{T^{-1}}$
     $\mathbf{T^{-1} = M^aL^b(LT^{-2})^c}$
     By identifying the two parts of the equation, we obtain : a = 0, b = $-\frac{1}{2}$ and c = $\frac{1}{2}$
     Therefore the pulsation : $\omega = \alpha\sqrt{\frac{g}{l}}$

2. Find the pulsation ω of a star knowing that it is a function of the density of the star ρ, its radius R and the constant of gravitation G.
   - $\omega = k\rho^a R^b G^c$ where k is a dimensionless numerical constant.
     Then, $[\omega] = [\rho]^a[R]^b[G]^c = \mathbf{T^{-1}}$
     $\mathbf{T^{-1} = (ML^{-3})^aL^b(M^{-1}L^3T^{-2})^c}$
     By identifying the two parts of the equation, we obtain : $a = c = \frac{1}{2}$ $and\ b = 0$
     Therefore, $\omega = k\sqrt{G\rho}$

3. The universal gravitational force is given by the relation : $F = G\frac{mM}{r^2}$
   - Determine the dimension of the constant G and its unit.
   - $F = G\frac{mM}{r^2} \rightarrow G = \frac{Fr^2}{mM} \rightarrow [G] = \frac{[F][r]^2}{[m][M]} = \frac{(MLT^{-2})(L)^2}{M^2} = \mathbf{M^{-1}L^3T^{-2}}$
     Therefore, $G = 6.67\text{x}10^{-11}\ kg^{-1}m^3 s^{-2}$

4. Give the dimensions and SI units of the following physical quantities :
   The permittivity of vacuum ε_0, the permeability of vacuum μ_0 , Planck's constant **h** , Boltzmann's constant **k** and Stefan's constant $\boldsymbol{\sigma}$.
   - Using the electric force relationship : $F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2} \rightarrow \varepsilon_0 = \frac{1}{4\pi F}\frac{qQ}{r^2}$
     $[\varepsilon_0] = \frac{1}{[F]}\frac{[q][Q]}{[r]^2} = \frac{I^2T^2}{MLT^{-2}L^2} = \mathbf{M^{-1}L^{-3}T^4\ I^2}$ ($kg^{-1}m^{-3}s^4A^2$)
   - Using the magnetic force relationship : $F = \mu_0\frac{I_1I_2l}{2\pi r} \rightarrow \mu_0 = \frac{2\pi Fr}{I_1I_2l}$
     $[\mu_0] = \frac{[F][r]}{[I_1][I_2][l]} = \frac{MLT^{-2}L}{I^2L} = \mathbf{MLT^{-2}\ I^{-2}}$ ($kgms^{-2}A^{-2}$)
   - Using the radiation energy relationship : $E = \boldsymbol{h}\nu \rightarrow \boldsymbol{h} = \frac{E}{\nu}$
     $[\boldsymbol{h}] = \frac{[E]}{[\nu]} = \mathbf{ML^2T^{-2}T} = \mathbf{ML^2T^{-1}}$ (kgm^2s^{-1}) or ($J \cdot s$)
   - Using the relationship for the kinetic energy of a monochromatic gas molecule :
     $E = \frac{3}{2}\mathbf{k}T \rightarrow \mathbf{k} = \frac{2E}{3T}$
     $[\mathbf{k}] = \frac{[E]}{[T]} = \frac{ML^2T^{-2}}{\theta} = \mathbf{ML^2T^{-2}\Theta^{-1}}$ ($kgm^2s^{-2}K^{-1}$) or ($J\ K^{-1}$)

- Using the relationship of radiated power per unit area of a black body :
  $\mathrm{P} = \frac{2\pi^4}{15}\frac{\mathbf{k}^4}{\mathrm{c}^2\mathrm{h}^3}\mathrm{T}^4 = \sigma T^4 \rightarrow \boldsymbol{\sigma} = \frac{2\pi^4}{15}\frac{\mathbf{k}^4}{\mathrm{c}^2\mathrm{h}^3}$
  $[\boldsymbol{\sigma}] = \frac{[\mathbf{k}]^4}{[c]^2[h]^3} = \frac{(ML^2T^{-2}\theta^{-1})^4}{(LT^{-1})^2(\mathrm{ML}^2T^{-1})^3} =$ $\mathbf{MT^{-3}\Theta^{-4}}$ (kg$s^{-3}K^{-4}$)

5. Give the dimensions and SI units of the following electrical quantities : RC, $\frac{\mathrm{L}}{\mathrm{R}}$ and $\sqrt{\mathrm{LC}}$
   - From the table, we have : [R] = $\mathbf{ML^2T^{-3}I^{-2}}$, [C] = $\mathbf{M^{-1}L^{-2}T^4I^2}$ and [L] = $\mathbf{ML^2T^{-2}I^{-2}}$
     Therefore, $[\mathrm{RC}] = \left[\frac{\mathrm{L}}{\mathrm{R}}\right] = \left[\sqrt{\mathrm{LC}}\right]$ = **T** (s)

6. Give the dimension and the unit of the constant R of ideal gases.
   - $PV = nRT \rightarrow R = \frac{PV}{nT}$
     $[R] = \frac{[P][V]}{[n][T]} = \frac{ML^{-1}T^{-2}L^3}{N\theta} =$ $\mathbf{ML^2T^{-2}N^{-1}\Theta^{-1}}$ (kg$m^2s^{-2}mol^{-1}K^{-1}$)

7. The speed of an object is expressed by the equation : $v = At^3 - Bt + \sqrt{C}$.
   - Give the dimensions and units of the coefficients A, B and C.
   - $[v] = [A][t]^3 = [B][t] = [C]^{1/2}$= $\mathbf{LT^{-1}}$
     [A] = $\mathbf{LT^{-4}}$ (ms^{-4}),
     [B] = $\mathbf{LT^{-2}}$ (ms^{-2})
     [C] = $\mathbf{L^2T^{-2}}$ (m^2s^{-2})

8. The force of friction acting on a body is proportional to the square of its speed.
   - Give the dimension and the unit of the constant of proportionality.
   - $F = kv^2 \rightarrow k = \frac{F}{v^2}$
     $[\,k] = \frac{[F]}{[v]^2} = \frac{MLT^{-2}}{(LT^{-1})^2} =$ $\mathbf{ML^{-1}}$ (kgm^{-1})

9. In an electric circuit, the intensity of the current obeys the differential equation :
   $R\frac{di}{dt} + \frac{1}{C}i = 0$ where R is a resistor and C is a capacitance.
   - Using these quantities, define τ a time homogeneous quantity.
   - $\left[R\frac{di}{dt}\right] = \left[\frac{1}{C}i\right] \rightarrow [R][C] = \frac{[i]}{[\frac{di}{dt}]} = \boldsymbol{T}$
     Therefore, $\tau = [\mathrm{RC}] = \mathbf{T}$

10. The elongation of a spring obeys the differential equation : $\frac{d^2y}{dt^2} + \frac{\omega_0}{Q}\frac{dy}{dt} + {\omega_0}^2 y = 0$
    - Determine the dimensions of the quantities ω_0 and Q.
    - $\left[\frac{d^2y}{dt^2}\right] = \left[\frac{\omega_0}{Q}\frac{dy}{dt}\right] = [{\omega_0}^2 y] = \frac{[y]}{T^2} = LT^{-2}$
      $[\omega_0]^2[y] = LT^{-2} \rightarrow [\omega_0] = \boldsymbol{T}^{-1}$ (s^{-1})
      $\frac{[\omega_0]}{[Q]}\left[\frac{dy}{dt}\right] = \frac{[\omega_0]}{[Q]}\frac{[y]}{[T]} = LT^{-2} \rightarrow [Q] = 1$ is a dimensionless quantity.

1.2 Calculation of Uncertainties:

Any physical measurement is accompanied by a margin of error called uncertainty. In general, if we measure a quantity **X** and we obtain an average value $\mathbf{a_0}$ with an uncertainty $\mathbf{\Delta a}$, we note : $\mathbf{X} = \mathbf{a_0} \pm \mathbf{\Delta a}$.
In this case $\mathbf{\Delta a}$ is called absolute uncertainty and has the same unit as $\mathbf{a_0}$.
One also defines the relative uncertainty $\frac{\mathbf{\Delta a}}{\mathbf{a}}$ which represents the importance of the error compared to the quantity measured . It is necessarily unit less and often given as a percentage.
Suppose that a physical quantity $G = G(x, y, z)$ depends on several quantities x, y, z measured with the uncertainties Δx, Δy, Δz.
The maximum possible error on G is : $\Delta \boldsymbol{G} = \left|\frac{\partial G}{\partial x}\right| \Delta \boldsymbol{x} + \left|\frac{\partial G}{\partial y}\right| \Delta \boldsymbol{y} + \left|\frac{\partial G}{\partial z}\right| \Delta \boldsymbol{z}$

Here are some cases to consider :

- $G = x \pm \mathrm{y} \rightarrow \Delta G = \Delta \boldsymbol{x} + \Delta \boldsymbol{y}$
- $G = xy \rightarrow \Delta G = |x|\Delta y + |y|\Delta x$
- $G = \frac{x}{y} \rightarrow \Delta G = \frac{|x|\Delta y + |y|\Delta x}{y^2}$
- $G = k\frac{x^{\alpha}y^{-\beta}}{(z+t)^{\gamma}} \rightarrow \Delta G = G(|\alpha|\frac{\Delta x}{x} + |\beta|\frac{\Delta y}{y} + |\gamma|\frac{\Delta z}{(z+t)} + |\gamma|\frac{\Delta t}{(z+t)})$

The number of significant digits retained in a result should never imply greater precision than the data.
In a calculation, the uncertainty of a result should never be greater than that of the least precise data.

Exercises :

1. Find the electrical resistance R in a simple electrical circuit where the current intensity has the value of I = 0.10 mA with an uncertainty ΔI = 0.01 mA and the voltage U across the resistor has the value of U = 1.50 V with an uncertainty ΔU = 0.01 V.
    - First of all, we calculate the average value of the resistance R knowing that R=U/I ; this gives : R = 1.50/0.10x10^{-3} = 15000 Ω.

      Then, the uncertainty $\Delta R = \frac{U\Delta I + I\Delta U}{I^2} = \frac{1.5*0.01*10^{-3}+0.1*10^{-3}*0.01}{(0.1*10^{-3})^2} = 1600\ \Omega$

      Finally, the measurement we made gives us : R = (15000 ± 1600) Ω.

2. The period of oscillation T of a simple pendulum is given by the formula : $T = 2\pi\sqrt{\frac{l}{g}}$.

   Give the uncertainty on g if the period is T= (2.20 ± 0.01) s and the length l = (120 ± 1) cm.
    - First we calculate the average value of the acceleration of gravity g :

      $T = 2\pi\sqrt{\frac{l}{g}} \quad \rightarrow \quad g = 4\pi^2\frac{l}{T^2} = 4\pi^2\frac{1.2}{(2.2)^2} \approx 9.79\ m/s^2$

      Then, the uncertainty $\Delta g = 4\pi^2[\frac{\Delta l}{T^2} + \frac{2l\Delta T}{T^3}] = 4\pi^2[\frac{0.01}{2.2^2} + \frac{2*1.2*0.01}{2.2^3}] \approx 0.17\ m/s^2$

      Finally, the acceleration of gravity : g = (9.79 ± 0.17) m/s ^{2}

3. Let A be the angle of a prism and D the angle of deviation of an incident ray after passing through the prism. The refractive index of the prism in the case of minimum deviation is :

   $n = \frac{\sin((A+D)/2)}{\sin(\frac{A}{2})}$.
    - Find the relative uncertainty $\frac{\Delta \mathbf{n}}{\mathbf{n}}$ in the case ΔA = ΔD.
    - $\Delta n = \left|\frac{\partial n}{\partial A}\right|\Delta A + \left|\frac{\partial n}{\partial D}\right|\Delta D$

      $\frac{\partial n}{\partial A} = \frac{1}{2}\left(\frac{\cos(\frac{A+D}{2})}{\sin(\frac{A}{2})} - cotan\left(\frac{A}{2}\right)\frac{sin(\frac{A+D}{2})}{\sin(\frac{A}{2})}\right) = \frac{1}{2}n\left(cotan(\frac{A+D}{2}) - cotan\left(\frac{A}{2}\right)\right)$

      $\frac{\partial n}{\partial D} = \frac{1}{2}\left(\frac{\cos(\frac{A+D}{2})}{\sin(\frac{A}{2})}\right) = \frac{1}{2}n\left(cotan(\frac{A+D}{2})\right)$

      $\frac{\Delta n}{n} = \frac{1}{2}(\left|cotan(\frac{A+D}{2}) - cotan\left(\frac{A}{2}\right)\right|\Delta A + \left|cotan(\frac{A+D}{2})\right|\Delta D)$

      In the case: ΔA = ΔD = ε, the relative uncertainty $\frac{\Delta \mathbf{n}}{\mathbf{n}}$ becomes:

      $\frac{\Delta n}{n} = \frac{1}{2}\varepsilon(\left|cotan\left(\frac{A+D}{2}\right) - cotan\left(\frac{A}{2}\right)\right| + \left|cotan\left(\frac{A+D}{2}\right)\right|)$

      Because $\left|cotan\left(\frac{A+D}{2}\right) - cotan\left(\frac{A}{2}\right)\right| = cotan\left(\frac{A}{2}\right) - cotan\left(\frac{A+D}{2}\right)$

      Therefore, $\frac{\Delta n}{n} = \frac{1}{2}\varepsilon(cotan\left(\frac{A}{2}\right))$

4. We measure the diameter and the mass of a gold ball :

   d = (10.00 ± 0.01) mm and m = (9.9 ± 0.1) g
    - Calculate the volume of the ball with its relative uncertainty as well as its absolute uncertainty.

- Calculate the density of the ball with its relative uncertainty as well as its absolute uncertainty.

- The volume of the ball $V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3 \approx 523.6\, mm^3 = 0.5236\, cm^3$
  $\frac{\Delta V}{V} = 3\frac{\Delta d}{d} = 0.003 = 0.3\% \rightarrow \Delta V = V * \frac{\Delta V}{V} = 523.6 * 0.003 \approx 1.571 \approx 1.6\, mm^3$
  Therefore, the volume : V = (523.6 ± 1.6) mm^3
- $\rho = \frac{m}{V} = \frac{9.9}{0.5236} \approx 18.91\, g/cm^3$
  $\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V} = \frac{0.1}{9.9} + 0.003 \approx 0.01 + 0.003 = 0.013 = 1.3\%$
  $\Delta\rho = \rho\Delta\rho = 18.91 * 0.013 \approx 0.25\, g/cm^3$
  Therefore, the density : $\rho = (18.91 \pm 0.25)\, g/cm^3$

5. The dimensions of a rectangle are : a = (5.35 ± 0.05) cm and b = (3.45 ± 0.04) cm
   - Calculate the perimeter and area of the rectangle.
   - L = 2(a+b) = 2(5.35 + 3.45) = 17.60 cm
     ΔL = 2(Δa + Δb) = 2*(0.05 + 0.04) = 0.18 ≈ 0.2 cm
     Therefore, the perimeter : L = (17.60 ± 0.18) cm or (17.6 ± 0.2) cm
     S = a*b = 5.35*3.45 = 18.4575 ≈ 18.46 cm^2
     ΔS = a*Δb + b*Δa = 5.35*0.04 + 3.45*0.05 = 0.3865 ≈ 0.39 cm^2
     Therefore, the area : S = (18.46 ± 0.39) cm^2 or (18.5 ± 0.4) cm^2

6. The radius of a sphere is r = (10.00 ± 0.08) cm
   - Calculate the area and volume of the sphere
   - $S = 4\pi r^2 = 4\pi 10^2 \approx 1256\, cm^2$
     $\Delta S = 8\pi r\Delta r = 8\pi 10 * 0.08 \approx 20\, cm^2$
     Therefore, the area : S = (1256 ± 20) cm^2
   - $V = \frac{4}{3}\pi r^3 \approx 4189\ cm^3$
     $\Delta V = 4\pi r^2 \Delta r \approx 100\, cm^3$
     Therefore, the volume V = (4189 ± 100) cm^3

7. A cylindrical volume with diameter d = (1.62 ± 0.03) cm and height h = (3.44 ± 0.05) cm has a mass m = (23.2 ± 0.1) g.
   - Calculate its volume and density.
   - $V = \frac{\pi}{4}d^2 h = \frac{\pi}{4}1.62^2 * 3.44 \approx 7.09\, cm^3$
     $\Delta V = \frac{\pi}{2}\, dh\Delta d + \frac{\pi}{4}d^2\Delta h \approx 0.37\ cm^3$
     Therefore, the volume : $V = (7.09 \pm 0.37)\, cm^3$
   - $\rho = \frac{m}{V} = \frac{23.2}{7.09} \approx 3.3\, g/cm^3$
     $\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V} = \frac{0.1}{23.2} + \frac{0.37}{7.09} \approx 0.06 \rightarrow \Delta\rho = 0.2\, g/cm^3$
     Therefore, the density : $\rho = (3.3 \pm 0.2)\, g/cm^3$

8. The sides opposite and adjacent to the angle θ of a right triangle are respectively :
   a = (12.1 ± 0.1) cm and b = (23.3 ± 0.2) cm.
   - Calculate the angle $\theta$and length of the hypotenuse.
   - $\theta = \tan^{-1}(\frac{\mathrm{a}}{\mathrm{b}}) = \tan^{-1}(\frac{12.1}{23.2}) \approx 27.4^0$

     $\Delta\theta = \frac{b}{a^2+b^2}\Delta a + \frac{a}{a^2+b^2}\Delta b \ \approx\ 0.0069\,rad \ \approx\ 0.4^0$

     Therefore, $\theta = 27.4^0 \ \pm\ 0.4^0$
   - $c = \sqrt{a^2+b^2} \approx 26.25$

     $\Delta c = \frac{a}{\sqrt{a^2+b^2}}\Delta a + \frac{b}{\sqrt{a^2+b^2}}\Delta b \ \approx 0.22$

     Therefore, the length of the hypotenuse is : c = (26.25 ± 0.22) cm

9. A vehicle consumes V = (48.6 ± 0.5) liters of fuel while traveling d = (530 ± 20) km.
   - Calculate its average consumption in liters per 100 km.
   - $C = \frac{100V}{d} = \frac{48.5*100}{530} \approx 9.2\ l/100km$

     $\Delta C = 100\frac{V\Delta d + d\Delta V}{d^2} \approx 0.5\ l/100km$

     Therefore, the average consumption in liters per 100 km is :

     C = (9.2 ± 0.5) $l/100\ km$

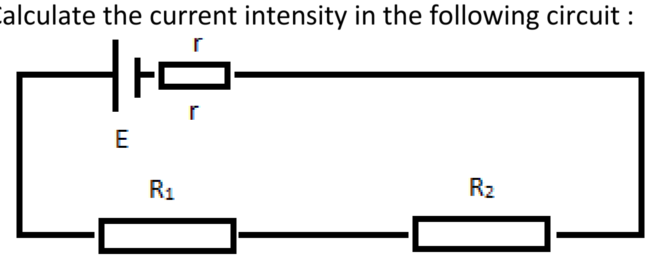
   The vehicle travels at a speed v = (100 ± 5) km/h and comes to a standstill in a time (3.3 ± 0.1) s.
   - Calculate the average acceleration of this vehicle in m/s^2 .

     (100 ± 5) km/h = (100 ± 5) * 1000/3600 m/s = [(1000 ± 50)/36] m/s
   - $a = dv/dt = \frac{1000/36}{3.3} \approx 8.4\ m/s^2$

     $\Delta a = \frac{t\Delta v + v\Delta t}{t^2} \approx 0.7\ m/s^2$

     Therefore, the average acceleration : a = (8.4 ± 0.7) m/s^2

10. Consider the following two circuits with :
    R_1 = (150 ± 10) Ω, R_2 = (200 ± 12) Ω, E = (100 ± 5) V and r = (10 ± 1) Ω
    - Calculate the current intensity in the following circuit :



    - The intensity of the current is given by the formula : $I = \frac{E}{r+R}$

      The two resistors R_1 and R_2 are in series : $R_T = R_1 + R_2$ = 150 + 200 = 350 Ω

      $I = \frac{100}{10+350} \approx 0.278\ A$

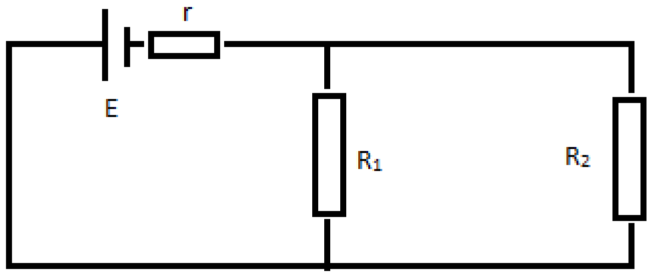
      $\Delta \mathrm{R_T} = \Delta\mathrm{R_1} + \Delta\mathrm{R_2} = 10 + 12 = 22\ \Omega$

      $\Delta\mathrm{I} = \left|\frac{\partial \mathrm{I}}{\partial \mathrm{E}}\right|\Delta\mathrm{E} + \left|\frac{\partial \mathrm{I}}{\partial \mathrm{r}}\right|\Delta\mathrm{r} + \left|\frac{\partial \mathrm{I}}{\partial \mathrm{R}}\right|\Delta\mathrm{R} = \frac{1}{\mathrm{r+R}}\Delta\mathrm{E} + \frac{\mathrm{E}}{(\mathrm{r+R})^2}\Delta\mathrm{r} + \frac{\mathrm{E}}{(\mathrm{r+R})^2}\Delta\mathrm{R}$

$\Delta I = \frac{1}{10+350}5 + \frac{100}{(10+350)^2}1 + \frac{100}{(10+350)^2}22 \approx 0.032\ A$

Therefore, the current intensity I = (0.278 ± 0.032) A

- Calculate the current intensity in the following circuit :



- The two resistors R_1 and R_2 are in parallel : $R_T = \frac{R_1 * R_2}{R_1 + R_2} = \frac{150*200}{150+200} \approx 85.714\ \Omega$

$$I = \frac{100}{10 + 85.714} \approx 1.045\ A$$

$$\Delta R_T = \frac{(R_1)^2 \Delta R_2 + (R_2)^2 \Delta R_1}{(R_1+R_2)^2} = \frac{150^2*12+200^2*10}{(150+200)^2} \approx 5.47\ \Omega$$

$$\Delta I = \left|\frac{\partial I}{\partial E}\right| \Delta E + \left|\frac{\partial I}{\partial r}\right| \Delta r + \left|\frac{\partial I}{\partial R}\right| \Delta R = \frac{1}{r+R}\Delta E + \frac{E}{(r+R)^2}\Delta r + \frac{E}{(r+R)^2}\Delta R$$

$$\Delta I = \frac{1}{10+85.714}5 + \frac{100}{(10+85.714)^2}1 + \frac{100}{(10+85.714)^2}5.47 \approx 0.123\ A$$

Therefore, the current intensity I = (1.045 ± 0.123) A

1.3 Vector Calculus:

A scalar is a quantity totally defined by a number and a unit : time, temperature, mass, energy, etc.
A vector is a mathematical entity defined by an origin, a direction, a direction and an intensity : displacement, velocity, acceleration, force, angular momentum, etc.

To each vector $\vec{\boldsymbol{A}}$, we can associate a unit vector $\vec{\boldsymbol{u}}$ which has the same direction and whose norm is equal to unity. The unit vector is obtained by dividing the initial vector by its modulus : $\vec{\boldsymbol{u}} = \frac{\vec{A}}{|\vec{A}|}$.

In the space referred to an orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, the vector $\vec{\boldsymbol{A}}$ is expressed by the formula :

$$\vec{\boldsymbol{A}} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$

and its modulus $|\vec{\boldsymbol{A}}|$ is : $|\vec{\boldsymbol{A}}| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$

Noting by $\alpha, \beta\ et\ \gamma$ the respective angles formed by the vector $\vec{\boldsymbol{A}}$ with the axes OX , OY and OZ :

$$A_x = Acos\alpha,\quad A_y = Acos\beta, \text{ and } A_z = cos\gamma.$$

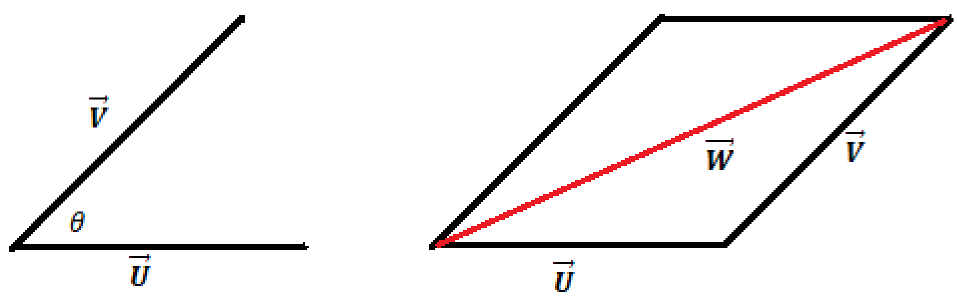
We can deduce the expression : $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

Vector Operations:

1.3.1 Addition of vectors:

Let be two vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$, the vector resultant of $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ is a vector $\vec{\boldsymbol{W}}$ located in the same plane such $\vec{\boldsymbol{W}} = \vec{\boldsymbol{U}} + \vec{\boldsymbol{V}}$ that :

$$\vec{\boldsymbol{W}} = (U_x + V_x)\vec{\imath} + (U_y + V_y)\vec{\jmath} + (U_z + V_z)\vec{k}$$



The magnitude of the vector $\vec{\boldsymbol{W}}$ is expressed by the expression :

$$|\vec{\boldsymbol{W}}| = \sqrt{|\vec{\boldsymbol{U}}|^2 + |\vec{\boldsymbol{V}}|^2 + 2|\vec{\boldsymbol{U}}| * |\vec{\boldsymbol{V}}| * cos\theta}$$

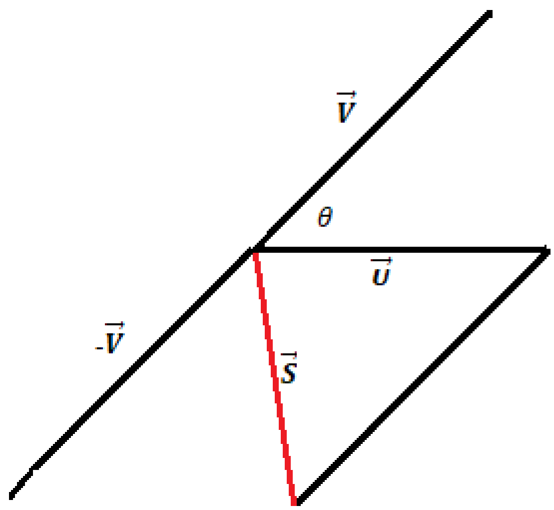
with $\theta = (\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}})$ the angle between the vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$.

If $\theta = 90^0$ we find the Pythagorean formula : $|\vec{\boldsymbol{W}}| = \sqrt{|\vec{\boldsymbol{U}}|^2 + |\vec{\boldsymbol{V}}|^2}$

The subtraction of two vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ is a vector $\vec{\boldsymbol{S}}$ located in the same plane such that :

$$\vec{\boldsymbol{S}} = \vec{\boldsymbol{U}} - \vec{\boldsymbol{V}} = (U_x - V_x)\vec{\imath} + (U_y - V_y)\vec{\jmath} + (U_z - V_z)\vec{k}.$$

The magnitude of the vector $\vec{\boldsymbol{S}}$ is expressed by the expression : $|\vec{\boldsymbol{S}}| = \sqrt{|\vec{\boldsymbol{U}}|^2 + |\vec{\boldsymbol{V}}|^2 - 2|\vec{\boldsymbol{U}}| * |\vec{\boldsymbol{V}}| * cos\theta}$



1.3.2 Scalar Product:

The dot product between two vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ is a scalar defined by the relation :

$$\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} = |\vec{\boldsymbol{U}}| * |\vec{\boldsymbol{V}}| * cos\theta$$

The scalar product is zero if one of the vectors is zero or the two vectors are orthogonal.

In the space referred to an orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, the scalar product is expressed by :

$$\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} = U_x \cdot V_x + U_y \cdot V_y + U_z \cdot V_z$$

The scalar product measures the angle between two vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$:

$$cos\theta = \frac{\vec{\boldsymbol{U}}.\vec{\boldsymbol{V}}}{|\vec{\boldsymbol{U}}||\vec{\boldsymbol{V}}|} = \frac{U_x \cdot V_x + U_y \cdot V_y + U_z \cdot V_z}{\sqrt{U_x{}^2 + U_y{}^2 + U_z{}^2}\sqrt{V_x{}^2 + V_y{}^2 + V_z{}^2}}$$

The angle is acute if the scalar product is positive and obtuse if the product is negative.

The vector projection of vector $\vec{\boldsymbol{U}}$ onto vector $\vec{\boldsymbol{V}}$ is $\vec{\boldsymbol{U}}_V = \frac{\vec{\boldsymbol{U}}.\vec{\boldsymbol{V}}}{|\vec{\boldsymbol{V}}|^2}\vec{\boldsymbol{V}}$

1.3.3 Vector Product:

The cross product between two vectors $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ is a vector denoted $\vec{\boldsymbol{W}} = \vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{V}}$ orthogonal to the two vectors with a direction that gives the triple $(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{W}})$a direct orientation. The vector product is zero if the vectors are collinear or if at least one of the two vectors is zero.

In space referred to an orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, the vector product is expressed by :

$$\vec{\boldsymbol{W}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ U_x & U_y & U_z \\ V_x & V_y & V_z \end{vmatrix} = (U_yV_z - U_zV_y)\vec{\imath} - (U_xV_z - U_zV_x)\vec{\jmath} + (U_xV_y - U_yV_x)\vec{k}$$

The modulus $|\vec{\boldsymbol{W}}| = |\vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{V}}| = |\vec{\boldsymbol{U}}| * |\vec{\boldsymbol{V}}| * |sin\theta|$ represents the area of the parallelogram constructed from the vectors $\vec{\boldsymbol{U}}$ et $\vec{\boldsymbol{V}}$.

1.3.4 Mixed Product:

The mixed product between three vectors $\vec{\boldsymbol{U}}$, $\vec{\boldsymbol{V}}$and $\vec{\boldsymbol{W}}$ is a scalar defined by the expression :

$$S = \vec{\boldsymbol{W}} \cdot (\vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{V}}) = \begin{vmatrix} W_x & W_y & W_z \\ U_x & U_y & U_z \\ V_x & V_y & V_z \end{vmatrix}$$

The mixed product is zero if the three vectors are coplanar, or two vectors are parallel, or if at least one of the three vectors is zero. It is positive if the triplet $(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{W}})$ forms a direct trihedron and negative if the triplet $(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{W}})$ forms an indirect one. The mixed product is invariant under circular permutation. Its absolute value represents the volume of the parallelepiped built on the three vectors $(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{W}})$.

1.3.5 The derivative:

The derived operation is an application which to each vector $\vec{\boldsymbol{A}}$ is associated a vector given by :

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\vec{\imath} + \frac{dA_y}{dt}\vec{\jmath} + \frac{dA_z}{dt}\vec{k}$$

1.3.6 Gradient:

The gradient operation is an application which, with each scalar field S, is associated with a vector field whose value at each point is given by :

$$\overrightarrow{\boldsymbol{grad}}S = \vec{\nabla}S = \frac{\partial S}{\partial x}\vec{\imath} + \frac{\partial S}{\partial y}\vec{\jmath} + \frac{\partial S}{\partial z}\vec{k}$$

1.3.7 Divergence:

The divergence operation is an application which to each vector $\vec{\boldsymbol{A}}$ is associated a scalar whose value at any point is given by :

$$div\vec{\boldsymbol{A}} = \vec{\nabla}.\vec{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

it is useful to know that : $\frac{\partial}{\partial t} div\vec{\boldsymbol{A}} = div\frac{\partial \vec{\boldsymbol{A}}}{\partial t}$ and $divS\vec{\boldsymbol{A}} = \vec{\nabla}S.\vec{\boldsymbol{A}} + Sdiv\vec{\boldsymbol{A}}$

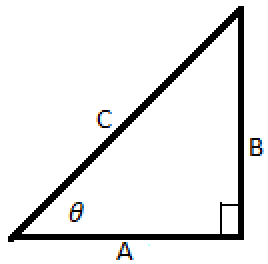
1.3.8 Rotational:

The rotational operation is an application which to each vector $\vec{\boldsymbol{A}}$ is associated a vector field whose value at each point is given by:

$$\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{A}} = \vec{\nabla} \wedge \vec{\boldsymbol{A}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\vec{\imath} - (\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z})\vec{\jmath} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\vec{k}$$

it is useful to know that : $\overrightarrow{\boldsymbol{rot}}\overrightarrow{\boldsymbol{grad}}M = \vec{\nabla} \wedge \vec{\nabla}M = 0$ and $div\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{A}} = \vec{\nabla}.\vec{\nabla} \wedge \vec{\boldsymbol{A}} = 0$

In a rectangular triangle, we have the following usual relations :



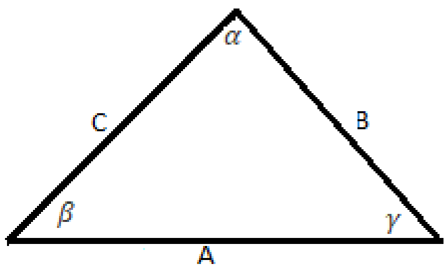
$$\sin\theta = \frac{A}{C}$$

$$\cos\theta = \frac{B}{C}$$

$$\tan\theta = \frac{B}{A}$$

$$\operatorname{cotan}\theta = \frac{A}{B}$$

In any triangle, we have the following relations of sines and cosines :



$$\frac{sin\alpha}{A} = \frac{sin\beta}{B} = \frac{sin\gamma}{C}$$

$$C = \sqrt{A^2 + B^2 - 2A * B * cos\gamma}$$

The sum of the interior angles of any triangle is equal to 180^0.

The area of a triangle is equal to : $S = \frac{1}{2}A \cdot Bsin\gamma = \frac{1}{2}A \cdot Csin\beta = \frac{1}{2}B \cdot Csin\alpha$

The sum of any two sides in a triangle is greater than the third one.

Exercises :

1. In an orthonormal reference OXYZ , we have the vectors :

   $\vec{\boldsymbol{A}} = 4\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$, $\vec{\boldsymbol{B}} = 2\vec{\imath} - 3\vec{\jmath} + 3\vec{k}$ and $\vec{\boldsymbol{C}} = -2\vec{\imath} - \vec{\jmath} + \vec{k}$

   - Calculate the modulus of each vector.
   - Calculate vectors $\vec{\boldsymbol{D}} = 2\vec{\boldsymbol{A}} + 3\vec{\boldsymbol{B}} - \vec{\boldsymbol{C}}$ and $\vec{\boldsymbol{E}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} - \vec{\boldsymbol{C}}$
   - Calculate the vector projection of the vector $\vec{\boldsymbol{D}}$ on the vector$\vec{\boldsymbol{E}}$

   - $|\vec{\boldsymbol{A}}| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2} = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$
   - $|\vec{\boldsymbol{B}}| = \sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2} = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$
   - $|\vec{\boldsymbol{C}}| = \sqrt{{C_x}^2 + C^2 + {C_z}^2} = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$
   - $\vec{\boldsymbol{D}} = 2\vec{\boldsymbol{A}} + 3\vec{\boldsymbol{B}} - \vec{\boldsymbol{C}} = 16\vec{\imath} - 14\vec{\jmath} + 12\vec{k}$
   - $\vec{\boldsymbol{E}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} - 2\vec{\boldsymbol{C}} = 10\vec{\imath} - 4\vec{\jmath} - 3\vec{k}$
   - $\vec{\boldsymbol{D}}_E = \frac{\vec{\boldsymbol{D}}.\vec{\boldsymbol{E}}}{|\vec{\boldsymbol{E}}|^2}\vec{\boldsymbol{E}} = \frac{16*10+14*4-12*3}{10^2+4^2+3^2}\vec{\boldsymbol{E}} = \frac{36}{25}\vec{\boldsymbol{E}}$

2. In the previous exercise :
   - Calculate the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$
   - Calculate the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{C}}$
   - Calculate the angle between $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$

   - $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = 4 * 2 + (-3)(-3) + 2 * 3 = 23$
   - $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = |\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}| \cos(a, B) \to \cos(a, b) = \frac{\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}}}{|\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}|} = \frac{23}{\sqrt{29}\sqrt{22}} \approx 0.91 \to \widehat{(a, b)} \approx 24.41^0$
   - $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}} = 4 * (-2) + (-3)(-1) + 2 * 1 = -3$
   - $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}} = |\vec{\boldsymbol{A}}||\vec{\boldsymbol{C}}| \cos(a, c) \to \cos(a, c) = \frac{\vec{\boldsymbol{A}}.\vec{\boldsymbol{C}}}{|\vec{\boldsymbol{A}}||\vec{\boldsymbol{C}}|} = \frac{-3}{\sqrt{29}\sqrt{6}} \approx -0.23 \to \widehat{(a, c)} \approx 103.15^0$
   - $\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}} = 2 * (-2) + (-3)(-1) + 3 * 1 = 2$
   - $\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}} = |\vec{\boldsymbol{B}}||\vec{\boldsymbol{C}}| \cos(b, c) \to \cos(b, c) = \frac{\vec{\boldsymbol{B}}.\vec{\boldsymbol{C}}}{|\vec{\boldsymbol{B}}||\vec{\boldsymbol{C}}|} = \frac{2}{\sqrt{22}\sqrt{6}} \approx 0.17 \to \widehat{(b, c)} \approx 79.98^0$

3. In the previous exercise, calculate :
   - The area of the parallelogram constructed from $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{E}}$.
   - the volume of the parallelepiped built on the three vectors $\vec{\boldsymbol{A}}, \vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{E}}$.

   - $S = |\vec{\boldsymbol{D}} \wedge \vec{\boldsymbol{E}}| = \left\| \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 16 & -14 & 12 \\ 10 & -4 & -3 \end{vmatrix} \right\| = |90\vec{\imath} + 168\vec{\jmath} + 74\vec{k}| = \sqrt{90^2 + 168^2 + 174^2}$

     Therefore, the area $S \approx 258\ u^2$
   - $V = |\vec{\boldsymbol{A}}.(\vec{\boldsymbol{D}} \wedge \vec{\boldsymbol{E}})| = \left\| \begin{vmatrix} 4 & -3 & 2 \\ 16 & -14 & 12 \\ 10 & -4 & -3 \end{vmatrix} \right\|$

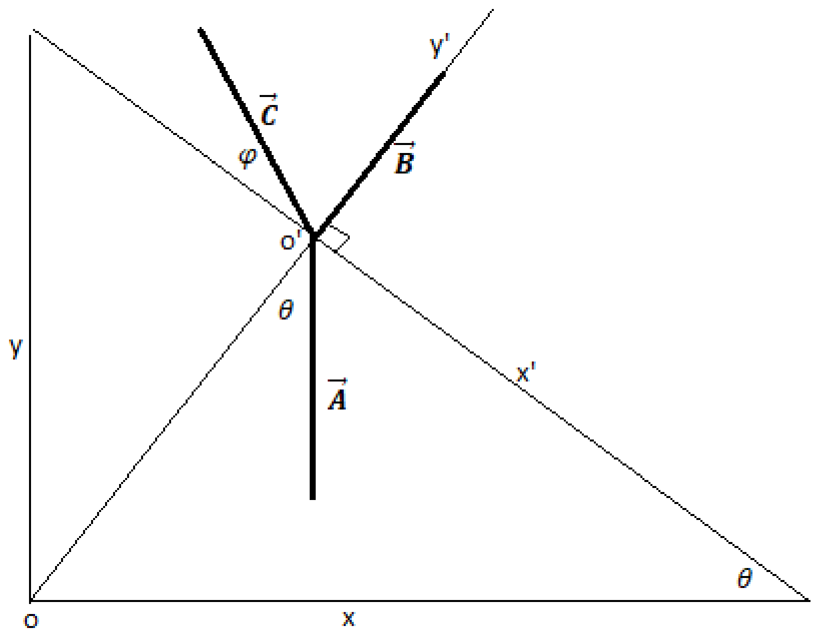
Therefore, The volume $V = 8\,u^3$

4. Let the vectors be $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$.
    - Demonstrate that $|\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}}|^2 + (\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}})^2 = |\vec{\boldsymbol{A}}|^2|\vec{\boldsymbol{B}}|^2$
    - Show that $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$ if $|\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}| = |\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}|$
    - $|\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}}| = \left||\vec{\boldsymbol{A}}| * |\vec{\boldsymbol{B}}| * sin\theta\right|$
      $\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} = |\vec{\boldsymbol{A}}| * |\vec{\boldsymbol{B}}| * cos\theta$

   Therefore : $|\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}}|^2 + (\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}})^2 = |\vec{\boldsymbol{A}}|^2|\vec{\boldsymbol{B}}|^2(cos^2\theta + sin^2\theta) = |\vec{\boldsymbol{A}}|^2|\vec{\boldsymbol{B}}|^2$
    - $|\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}| = |\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}| \rightarrow |\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}|^2 = |\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}|^2 = (\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}).(\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}) = (\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}).(\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}})$
      $\rightarrow |\vec{\boldsymbol{A}}|^2 + |\vec{\boldsymbol{B}}|^2 + 2\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} = |\vec{\boldsymbol{A}}|^2 + |\vec{\boldsymbol{B}}|^2 - 2\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} \rightarrow \vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} = -\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} \rightarrow \vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} = 0$

   Therefore : $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$

5. Let be the three vectors $\vec{\boldsymbol{A}},\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$. Give their projections on the axes OX, OY, O'X' and O'Y' knowing that $|\vec{\boldsymbol{A}}| = 10\,u, |\vec{\boldsymbol{B}}| = 8\,u\ and\ |\vec{\boldsymbol{C}}| = 9\,u, \theta = 60^0$ and $\varphi = 15^0$.
    - Calculate the resultant $\vec{\boldsymbol{D}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}$ in the system OXY and O'X'Y'.



	OX	YY	O'X'	O'Y'
$\vec{\boldsymbol{A}}$	0	-10	10*sin60 ≈ 8.66	-10*cos60= -5
$\vec{\boldsymbol{B}}$	8*sin60 ≈6.93	8*cos60= 4	0	8
$\vec{\boldsymbol{C}}$	-9*c sin15 ≈2.33	9* cos15 ≈8.69	-9*cos15 ≈ -8.69	9*sin15 ≈2.33
$\vec{\boldsymbol{D}}$	4.60	2.69	-0.03	5.33

Note: $|\vec{\boldsymbol{D}}| \approx \sqrt{28.40}\,u$ in both systems.

6. Let the vector $\vec{\boldsymbol{A}} = 4xt^2\vec{\imath} - 3\sin(2t)\vec{\jmath} + \frac{2x}{t+5}\vec{k}$ where t and x are independent real variables. Calculate $\frac{d\vec{\boldsymbol{A}}}{dt}$ and $\frac{d\vec{A}}{dx}$.
   - $\frac{d\vec{A}}{dt} = 8xt\vec{\imath} - 6cos(2t)\vec{\jmath} - \frac{2x}{(t+5)^2}\vec{k}$
   - $\frac{d\vec{A}}{dx} = 4t^2\vec{\imath} + \frac{2}{t+5}\vec{k}$

7. Let the functions $f(x,y,z)$ and be $g(x,y,z)$ defined in the space of reals as :
   - $f(x,y,z) = \sqrt{x^2+y^2+z^2} = |\vec{\boldsymbol{r}}|$ and $g(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{|\vec{\boldsymbol{r}}|}$
   - Calculate $\overrightarrow{\boldsymbol{grad}}f = \vec{\nabla}f(x,y,z)$ and $\overrightarrow{\boldsymbol{grad}}g = \vec{\nabla}g(x,y,z)$
   - $\vec{\nabla}f(x,y,z) = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} = \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{\imath} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{\jmath} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$

     Therefore : $\vec{\nabla}|\vec{\boldsymbol{r}}| = \frac{x\vec{\imath}+y\vec{\jmath}+z\vec{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}$
   - $\vec{\nabla}g(x,y,z) = \frac{\partial g}{\partial x}\vec{\imath} + \frac{\partial g}{\partial y}\vec{\jmath} + \frac{\partial g}{\partial z}\vec{k}$

     $\vec{\nabla}g(x,y,z) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}\vec{\imath} + \frac{-y}{(x^2+y^2+z^2)^{3/2}}\vec{\jmath} + \frac{-z}{(x^2+y^2+z^2)^{3/2}}\vec{k}$

     Therefore : $\vec{\nabla}\frac{1}{|\vec{\boldsymbol{r}}|} = \frac{-(\overrightarrow{x\vec{\imath}+y\vec{\jmath}+z\vec{k}})}{(x^2+y^2+z^2)^{3/2}} = \frac{-\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^3}$

8. The vector $\vec{\boldsymbol{V}}(t)$ is dependent on the real variable t.
   - Prove that if its direction is constant, then $\left|\frac{d\vec{\boldsymbol{V}}}{dt}\right| = \frac{d|\vec{\boldsymbol{V}}|}{dt}$.
   - Show that if its magnitude $|\vec{\boldsymbol{V}}|$ is constant, then $\vec{\boldsymbol{V}} \perp \frac{d\vec{\boldsymbol{V}}}{dt}$

     The vector $\vec{\boldsymbol{V}}(t)$is expressed as $\vec{\boldsymbol{V}}(t) = |\vec{\boldsymbol{V}}|\vec{\boldsymbol{v}}$ where $\vec{\boldsymbol{v}}$ is the unit vector of $\vec{\boldsymbol{V}}(t)$

     $\frac{d\vec{\boldsymbol{V}}}{dt} = \frac{d|\vec{\boldsymbol{V}}|\vec{\boldsymbol{v}}}{dt} = |\vec{\boldsymbol{V}}|\frac{d\vec{\boldsymbol{v}}}{dt} + \vec{\boldsymbol{v}}\frac{d|\vec{\boldsymbol{V}}|}{dt}$
   - If the direction of $\vec{\boldsymbol{V}}$ is constant, then $\vec{\boldsymbol{v}}$ is constant $\rightarrow \frac{d\vec{\boldsymbol{v}}}{dt} = 0$

     Therefore : $\left|\frac{d\vec{\boldsymbol{V}}}{dt}\right| = \frac{d|\vec{\boldsymbol{V}}|}{dt}$
   - If the magnitude $|\vec{\boldsymbol{V}}|$ is constant: $|\vec{\boldsymbol{V}}|^2 = \vec{\boldsymbol{V}}.\vec{\boldsymbol{V}} =$ constant $\rightarrow \frac{d|\vec{\boldsymbol{V}}|^2}{dt} = 0 \rightarrow 2\vec{\boldsymbol{V}}.\frac{d\vec{\boldsymbol{V}}}{dt} = 0$

     Therefore : $\vec{\boldsymbol{V}} \perp \frac{d\vec{\boldsymbol{V}}}{dt}$

9. Let the vector $\vec{\boldsymbol{V}} = 8xy\vec{\imath} - 6ycos(2z)\vec{\jmath} - \frac{2x}{(z+5)}\vec{k}$
   - Calculate $\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{V}}$, $div\vec{\boldsymbol{V}}$ and $div\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{V}}$.
   - $\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{V}} = \vec{\nabla}\wedge\vec{\boldsymbol{V}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8xy & -6ycos(2z) & -\frac{2x}{(z+5)} \end{vmatrix} = -12ysin(2z)\vec{\imath} + \frac{2}{(z+5)}\vec{\jmath} - 8x\vec{k}$

- $div\vec{\boldsymbol{V}} = \vec{\nabla}.\vec{\boldsymbol{V}} = 8y - 6cos(2z) + \frac{2x}{(z+5)^2}$
- $div\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{V}} = 0$ (Theorem)

10. Let be the vector $\vec{\boldsymbol{r}} = acos\omega t\vec{\imath} + bsin\omega t\vec{\jmath}$ where a, b and ω are real constants.

    Calculate $\vec{r}\wedge\frac{d\vec{\boldsymbol{r}}}{dt}$ and $\vec{r}\wedge\frac{d^2\vec{\boldsymbol{r}}}{dt^2}$.

    - $\frac{d\vec{\boldsymbol{r}}}{dt} = -a\omega sin\omega t\vec{\imath} + b\omega acos\omega t\vec{\jmath}$

      $\rightarrow \frac{d^2\vec{\boldsymbol{r}}}{dt^2} = -a\omega^2 cos\omega t\vec{\imath} - \omega^2 bsin\omega t\vec{\jmath} = -\omega^2\vec{\boldsymbol{r}}$
    - $\vec{r}\wedge\frac{d\vec{\boldsymbol{r}}}{dt} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ acos\omega t & bsin\omega t & 0 \\ -a\omega sin\omega t & b\omega acos\omega t & 0 \end{vmatrix} = ab\omega\vec{k}$

      $\vec{r}\wedge\dfrac{d^2\vec{\boldsymbol{r}}}{dt^2} = 0$

Chapter 2

Kinematics

Kinematics is the branch of mechanics that describes the motion of a body with respect to time.
The trajectory is the geometric locus of the successive positions occupied by the material point over time and with respect to a chosen reference system.

2.1 Displacement:

Displacement is a vector linking two positions of the mobile M_1 and M_2 at times t_1 and t_2 respectively :

$$\overrightarrow{\mathbf{M_1M_2}} = \overrightarrow{\mathbf{OM_2}} - \overrightarrow{\mathbf{OM_1}}$$

2.2 Velocity:

The velocity vector $\vec{\mathbf{V}}$ of a mobile is the rate of variation of its position vector $\overrightarrow{\boldsymbol{OM}}$ with respect to time. This variation can concern the direction of $\overrightarrow{\boldsymbol{OM}}$, its modulus or both. The unit of velocity in the international system (SI) is the meter per second (m/s).
The average velocity of a mobile between two times t_1 and t_2 corresponding to positions M_1 and M_2 is defined by the ratio : $\vec{\mathbf{V}}_m = \frac{\overrightarrow{\mathbf{OM_2}} - \overrightarrow{\mathbf{OM_1}}}{\mathrm{t_2 - t_1}} = \frac{\Delta\overrightarrow{\boldsymbol{OM}}}{\Delta t}$

The instantaneous velocity vector of a mobile, at time t, is given by the relation :

$$\vec{\boldsymbol{V}} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{\boldsymbol{OM}}}{\Delta t} = \frac{d\overrightarrow{\boldsymbol{OM}}}{dt}$$

The instantaneous velocity vector $\vec{\boldsymbol{V}}$ is at each instant tangent to the trajectory and its direction is that of the movement.

2.3 Acceleration:

The acceleration vector $\vec{\boldsymbol{a}}$ translates the rate of variation of the velocity vector $\vec{\boldsymbol{V}}$ as a function of time. This variation can concern the direction of the velocity, its modulus or both. The unit of acceleration in the international system (SI) is m/s^2 .
The average acceleration of a mobile between two times t_1 and t_2 corresponding to positions M_1 and M_2 is defined by the ratio : $\vec{\mathbf{a}}_m = \frac{\vec{\mathrm{V}}_2 - \vec{\mathrm{V}}_1}{\mathrm{t_2 - t_1}} = \frac{\Delta\vec{\boldsymbol{V}}}{\Delta t}$

$\vec{\mathrm{V}}_1$ and $\vec{\mathrm{V}}_2$ are the velocities of the mobile at positions M_1 and M_2 .
The instantaneous acceleration vector of a mobile at time t is defined by the relation :

$$\vec{\boldsymbol{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\boldsymbol{V}}}{\Delta t} = \frac{d\vec{\boldsymbol{V}}}{dt}$$

$\vec{\boldsymbol{a}}$ is always oriented towards the concave side of the trajectory.
The position is defined from velocity by : $\vec{\boldsymbol{r}}(t) - \vec{\boldsymbol{r}}(0) = \int_0^t \vec{\boldsymbol{V}}(\tau)d\tau$
The velocity is defined from acceleration by : $\vec{\boldsymbol{V}}(t) - \vec{\boldsymbol{V}}(0) = \int_0^t \vec{\boldsymbol{a}}(\tau)d\tau$
The motion is said to be accelerated if $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{V}}$ is positive , and decelerated or delayed if $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{V}}$ is negative.
As for the direction of movement, it is indicated by the direction of the velocity $\vec{\boldsymbol{V}}$.

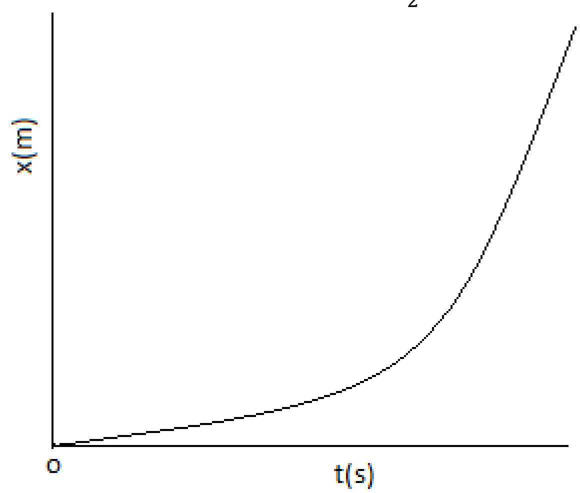
2.4 Straight motion:

Rectilinear movement is a movement for which the trajectory followed is straight. The reference can then be reduced to an origin O and an axis OX carried by the trajectory.

The position M of the mobile is identified by the position vector : $\overrightarrow{\boldsymbol{OM}}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t})\vec{\imath}$

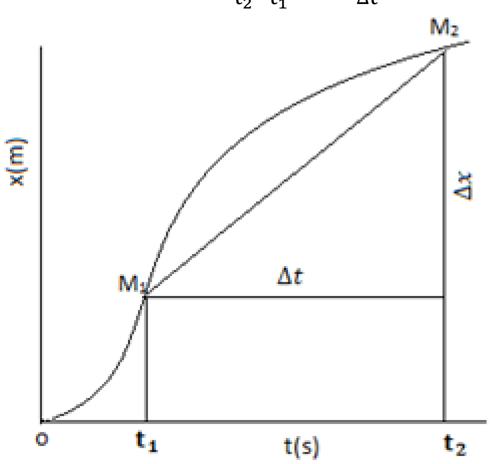
The graph of $\boldsymbol{x}(\boldsymbol{t})$ constitutes the diagram of the spaces.

Example : the diagram of the spaces in the case of a free fall of a body released at the origin O of a vertical axis directed downwards is : $\boldsymbol{x}(\boldsymbol{t}) = \frac{1}{2}gt^2$



Displacement is a vector linking two positions of the mobile M_1 and M_2 on the axis OX at times t_1 and t_2 respectively : $\overrightarrow{\mathbf{M_1M_2}} = \overrightarrow{\mathbf{OM_2}} - \overrightarrow{\mathbf{OM_1}} = (x_2 - x_1)\vec{\imath}$

The average velocity is $\vec{\mathbf{V}}_m = \frac{(x_2 - x_1)}{\mathrm{t_2 - t_1}}\vec{\imath} = \frac{\Delta x}{\Delta \mathrm{t}}\vec{\imath}$

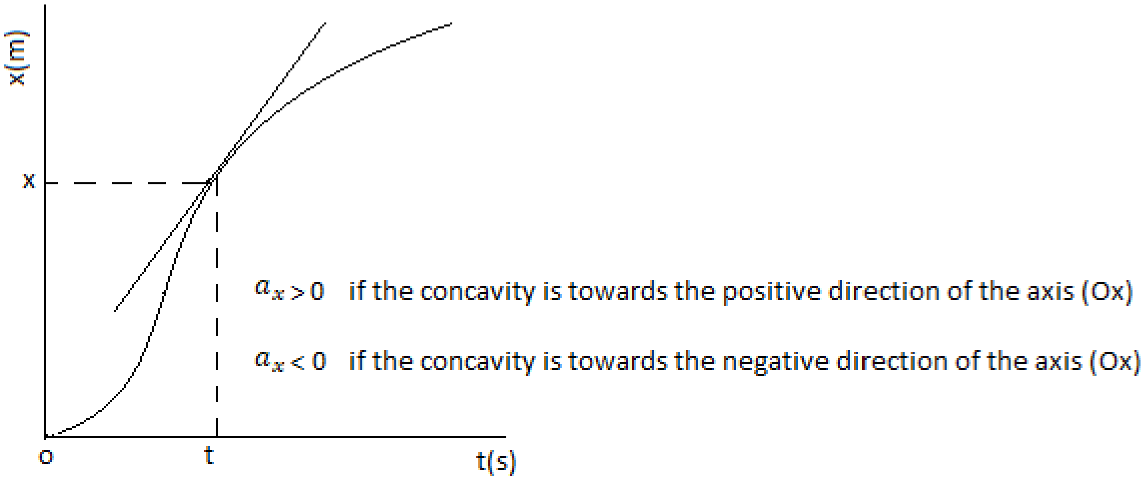


The average velocity is therefore the slope of the secant $M_1 M_2$.

the scalar average speed is given by the ratio $= \frac{\text{distance totale parcourue}}{\text{temps total mis}}$

The instantaneous velocity vector $\vec{\boldsymbol{V}} = V_x\vec{\imath} = \frac{dx}{\mathrm{dt}}\vec{\imath}$

where $\frac{dx}{\mathrm{dt}}$ represents the slope of the tangent to the space diagram at the point corresponding to time t.

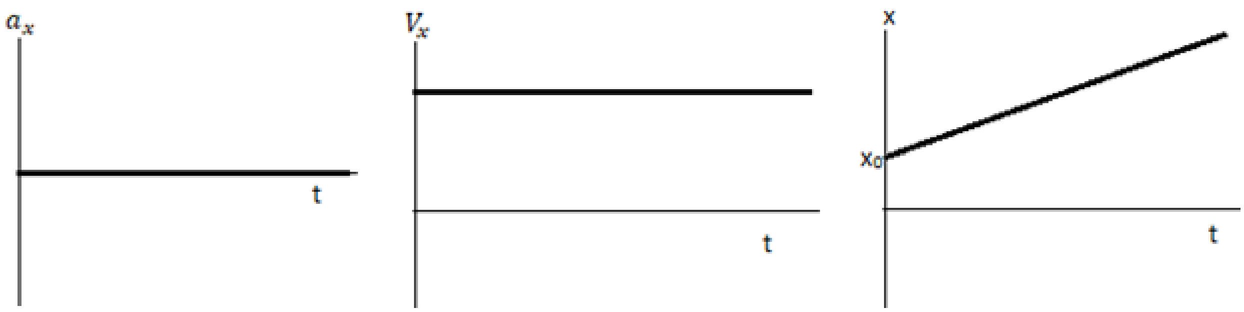


The instantaneous acceleration vector $\vec{\boldsymbol{a}} = a_x\vec{\imath} = \frac{dV_x}{\mathrm{dt}}\vec{\imath} = \frac{d^2x}{\mathrm{dt}^2}\vec{\imath}$

2.4.1 Uniform rectilinear motion:

The motion of a mobile is uniform when the algebraic value of its velocity V_x is constant. It is a motion without acceleration by virtue of the relation :

$$a_x = \frac{dV_x}{\mathrm{dt}} = 0 \rightarrow x(t) - x(0) = \int_0^t V_x d\tau = V_x t \;\rightarrow\; x(t) = x(0) + V_x t$$



2.4.2 Uniformly varied rectilinear motion:

The motion of a mobile is uniformly varied when the acceleration of the mobile a_x is constant.

$$a_x = \frac{dV_x}{\mathrm{dt}} \rightarrow V_x(t) - V_x(0) = \int_0^t a_x d\tau = a_x t \;\rightarrow\; V_x(t) = V_x(0) + a_x t$$

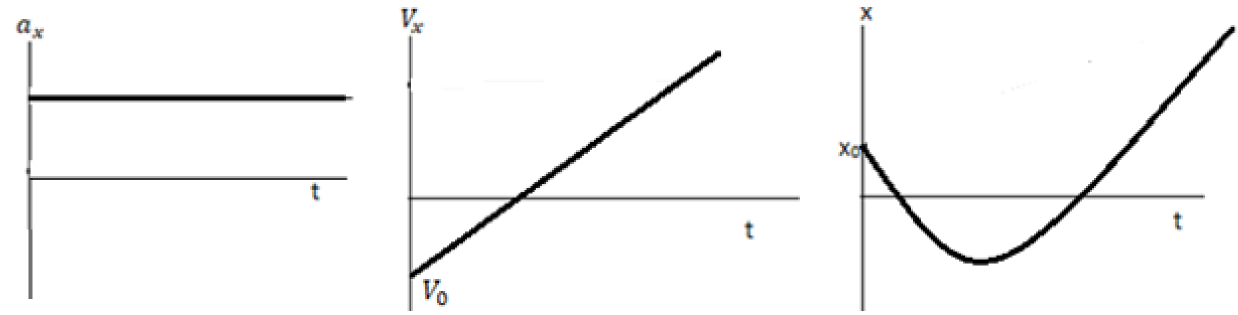
$$V_x = \frac{dx}{\mathrm{dt}} \rightarrow x(t) - x(0) = \int_0^t V_x(\tau) d\tau = \int_0^t (V_x(0) + a_x\tau) d\tau$$

$$x(t) = x(0) + V_x(0)t + \frac{1}{2}a_x t^2$$

$$x(t) = x(0) + V_x(t)t - \frac{1}{2}a_x t^2$$

$$x(t) = x(0) + \frac{V_x(0)+V_x(t)}{2}t$$

$$x(t) = x(0) + \frac{V_x(t)^2 - V_x(0)^2}{2a_x}$$



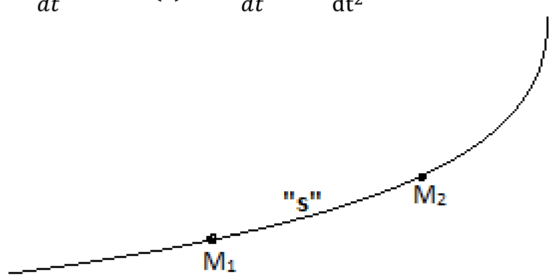
Particular natures of rectilinear motion :

- V_x constant : the movement is uniform rectilinear;
- a_x constant : the movement is rectilinear and uniformly varied;
- $a_x\, V_x > 0$: the movement is rectilinear accelerated (uniformly if a_x constant);
- $a_x\, V_x < 0$: the movement is rectilinear decelerated or delayed (uniformly if a_x constant).

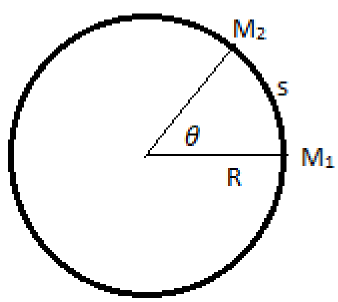
2.5 Curvilinear movement:

The abscissa "s" representing the algebraic value of the arc $\overleftrightarrow{M_1M_2}$ between two times t_1 and t_2 corresponding to positions M_1 and M_2 is introduced. We define, respectively, the curvilinear velocity and acceleration by the relations :

$V(t) = \frac{ds(t)}{dt}$ and $a(t) = \frac{dV(t)}{dt} = \frac{d^2s(t)}{\mathrm{dt}^2}$



Example : Varied circular motion on a trajectory of radius R :



$$s = R\theta$$
$$V = R\frac{d\theta}{dt}$$
$$a = R\frac{d^2\theta}{dt^2}$$

In general, to determine the curvilinear motion of a mobile, we use its intrinsic components which are its algebraic projections on :

- a tangential axis (MT) provided with the unit vector $\vec{\mathbf{u}}_T$ directed in the direction of movement
- a normal axis (MN) provided with the unit vector $\vec{\mathbf{u}}_N$, oriented towards the concave side of the trajectory.

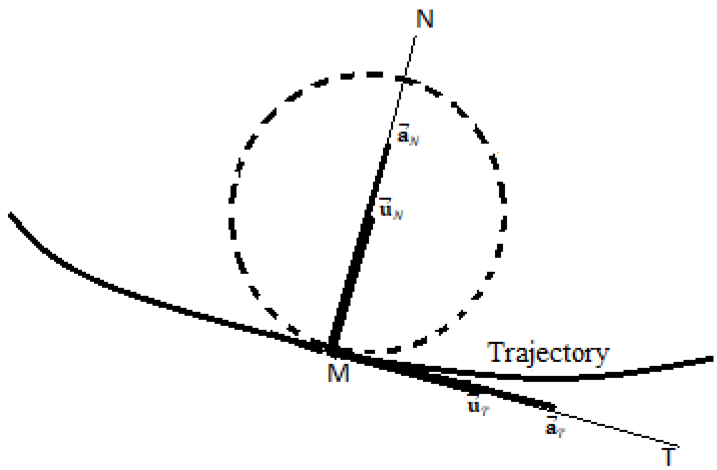
The velocity vector $\vec{\boldsymbol{v}}$ is oriented according to the vector $\vec{\mathbf{u}}_T$: $\vec{\boldsymbol{v}} = v\vec{\mathbf{u}}_T = \frac{ds}{dt}\vec{\mathbf{u}}_T$
The vector acceleration $\vec{\boldsymbol{a}}$ has two components : $\vec{\boldsymbol{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_N = a_T\vec{\mathbf{u}}_T + a_N\vec{\mathbf{u}}_N$
a_T and a_N are, respectively, the tangential and normal components of the acceleration.
the unit vectors $\vec{\mathbf{u}}_T$ and $\vec{\mathbf{u}}_N$ form an orthonormal basis called the *Frenet basis*.

$$\vec{\boldsymbol{a}} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}\vec{\mathbf{u}}_T + \frac{ds}{dt}\frac{d\vec{\mathbf{u}}_T}{dt} = \frac{d^2s}{dt^2}\vec{\mathbf{u}}_T + \frac{ds}{dt}\frac{ds}{dt}\frac{d\vec{\mathbf{u}}_T}{ds}$$

We will admit without proof that : $\frac{d\vec{\mathbf{u}}_T}{ds} = \frac{1}{\rho}\vec{\mathbf{u}}_N$
where ρ (s) is called the radius of curvature of the trajectory at the considered point.
This results in the following explicit expression of the acceleration :

$$\vec{\boldsymbol{a}} = \frac{dv}{dt}\vec{\mathbf{u}}_T + \frac{v^2}{\rho}\vec{\mathbf{u}}_N$$

The radius of curvature of the trajectory at the considered point is given by the expression : $\rho = \frac{v^3}{|\vec{v}\wedge\vec{a}|}$



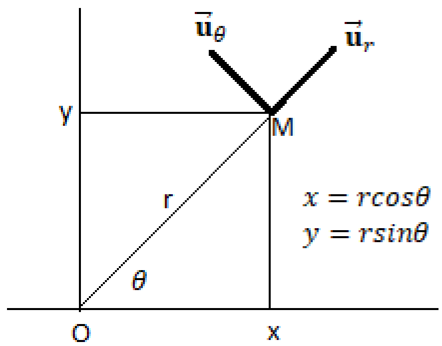
Varied rectilinear motion : rectilinear means that there is no variation in the direction of the velocity vector; in this case the radius of curvature ρ of the trajectory is infinite and therefore the normal component $a_N = 0$, and varied means the tangential component $a_T = \frac{dv}{dt} \neq 0$.
Uniform circular motion : circular means that the mobile moves on a circular trajectory of radius R = ρ and therefore $a_N = \frac{v^2}{\rho} \neq 0$, and uniform means $a_T = 0$.

<u>2.6 Study of motion in polar coordinates:</u>
This coordinate system is suitable for studying plane motions with rotational symmetry. The identification is carried out relative to a polar axis (OX), of origin O called pole. We can then locate the position of any point M of the plane containing (OX) by :
Polar radius $r(t) = |\overrightarrow{\boldsymbol{OM}}|$ and polar angle $\theta(t) = (\overrightarrow{\boldsymbol{OX}}, \overrightarrow{\boldsymbol{OM}})$
In this system we use the base constituted by the unit vectors :

- $\vec{\mathbf{u}}_r$ having the direction and sense of $\overrightarrow{\boldsymbol{OM}}$
- $\vec{\mathbf{u}}_\theta$ obtained by rotating $\vec{\mathbf{u}}_r$ through an angle 90^0 in the counterclockwise direction.

The base $(\vec{\mathbf{u}}_r, \vec{\mathbf{u}}_\theta)$ is related to the point M, and because of this the directions of the unit vectors can vary with time. Their derivatives satisfy a certain number of relations, in particular, it will be admitted without proof that :

$\frac{d\vec{\mathbf{u}}_r}{dt} = \frac{d\theta}{dt}\vec{\mathbf{u}}_\theta$ and $\frac{d\vec{\mathbf{u}}_\theta}{dt} = -\frac{d\theta}{dt}\vec{\mathbf{u}}_r$



The polar coordinates r and θ of the point M are related to the Cartesian coordinates by the following relations :

$x = rcos\theta$ and $y = rsin\theta$

$\vec{\mathbf{u}}_r = \vec{\imath}cos\theta + \vec{\jmath}sin\theta$ and $\vec{\mathbf{u}}_\theta = -\vec{\imath}sin\theta + \vec{\jmath}cos\theta$

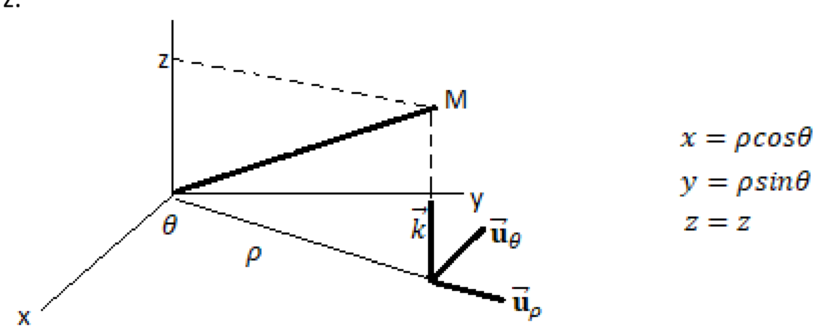
The position vector is : $\overrightarrow{\boldsymbol{OM}} = r\vec{\mathbf{u}}_r$

The velocity vector is : $\vec{\boldsymbol{V}} = \frac{dr}{dt}\vec{\mathbf{u}}_r + r\frac{d\theta}{dt}\vec{\mathbf{u}}_\theta = \dot{r}\vec{\mathbf{u}}_r + r\dot{\theta}\vec{\mathbf{u}}_\theta$

The acceleration vector is : $\vec{\boldsymbol{a}} = (\ddot{r} - r\dot{\theta}^2)\vec{\mathbf{u}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{\mathbf{u}}_\theta$

2.7 Study of movement in cylindrical coordinates:

When a movement takes place on a cylindrical or spiral surface, we often use the cylindrical coordinates which we define with respect to the Cartesian system. The mobile M is then identified in the base $(\vec{\mathbf{u}}_\rho, \vec{\mathbf{u}}_\theta, \vec{\boldsymbol{k}})$ by: the polar coordinates ρ and θ of its projection on the plane (O, X, Y) and its axial coordinate z.



$\vec{\mathbf{u}}_\rho = \vec{\imath}cos\theta + \vec{\jmath}sin\theta$ and $\vec{\mathbf{u}}_\theta = -\vec{\imath}sin\theta + \vec{\jmath}cos\theta$

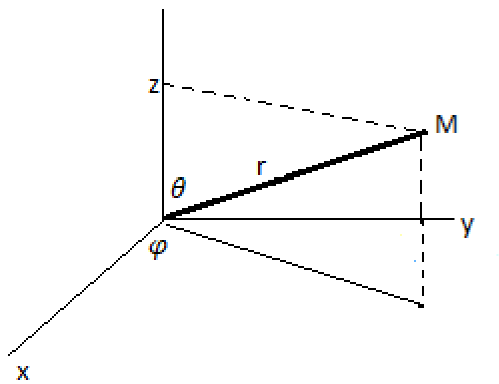
The position vector is : $\overrightarrow{\boldsymbol{OM}} = \rho\vec{\mathbf{u}}_\rho + z\vec{\boldsymbol{k}}$

The velocity vector is : $\vec{\boldsymbol{V}} = \frac{d\rho}{dt}\vec{\mathbf{u}}_\rho + \rho\frac{d\theta}{dt}\vec{\mathbf{u}}_\theta + \dot{z}\vec{\boldsymbol{k}} = \dot{\rho}\vec{\mathbf{u}}_\rho + \rho\dot{\theta}\vec{\mathbf{u}}_\theta + \dot{z}\vec{\boldsymbol{k}}$

The acceleration vector is : $\vec{\boldsymbol{a}} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{\mathbf{u}}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{\mathbf{u}}_\theta + \ddot{z}\vec{\boldsymbol{k}}$

2.8 Study of movement in spherical coordinates:

When a movement takes place on a spherical surface, one often uses the spherical coordinates which one defines compared to the Cartesian system. The mobile M is then identified in the $(\vec{\mathbf{u}}_{r,}, \vec{\mathbf{u}}_{\theta}, \vec{\mathbf{u}}_{\varphi})$ base by : (r, θ, φ) where r representing the radial coordinate corresponding to the distance from the origin O to the mobile M, the angle θ between 0 and π, corresponds to the angle between OM and the axis OZ and the angle φ between 0 and 2π, corresponds to the angle between the plane defined by the axis OZ and OM with the axis OX .



$x = rsin\varphi cos\theta$

$y = rsin\varphi sin\theta$

$z = rcos\varphi$

$$\vec{\mathbf{u}}_r = \vec{\imath}cos\varphi sin\theta + \vec{\jmath}sin\varphi sin\theta + \vec{k}cos\theta$$
$$\vec{\mathbf{u}}_\theta = \vec{\imath}cos\varphi cos\theta + \vec{\jmath}sin\varphi cos\theta - \vec{k}sin\theta$$
$$\vec{\mathbf{u}}_\varphi = -\vec{\imath}sin\varphi + \vec{\jmath}cos\varphi$$

The position vector is : $\overrightarrow{\boldsymbol{OM}} = r\vec{\mathbf{u}}_r$

The velocity vector is : $\vec{\boldsymbol{V}} = \frac{dr}{dt}\vec{\mathbf{u}}_r + r\frac{d\theta}{dt}\vec{\mathbf{u}}_\theta + r\dot{\varphi}sin\theta\vec{\mathbf{u}}_\varphi = \dot{r}\vec{\mathbf{u}}_r + r\dot{\theta}\vec{\mathbf{u}}_\theta + r\dot{\varphi}sin\theta\vec{\mathbf{u}}_\varphi$

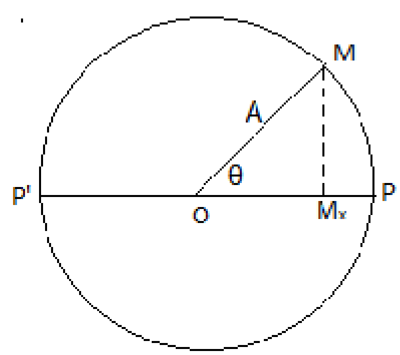
The acceleration vector is :

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)\vec{\mathbf{u}}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)\vec{\mathbf{u}}_\theta + \left(r\ddot{\varphi}sin\theta + 2\dot{r}\dot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta\right)\vec{\mathbf{u}}_\varphi$$

2.9 Simple harmonic motion:

Consider a point M moving on a circle of radius A at constant angular velocity $\omega = \frac{d\theta}{dt}$.

When M moves on its trajectory, its projection M_x on the axis (OX), performs oscillations on the segment PP' centered in O.



The projection M_x of the mobile M has for abscissa : $x = Acos\theta$

The value of the angle $\theta(t)$ is $\theta(t) = \theta_0 + \omega t$

Such that : $x = Acos(\theta_0 + \omega t)$

The movement of M_x is a simple harmonic motion; its characteristics are :

A : amplitude

ω : the pulsation or angular frequency

$\theta_0 + \omega t$: the phase and θ_0 the initial phase.

Harmonic motion is periodic :

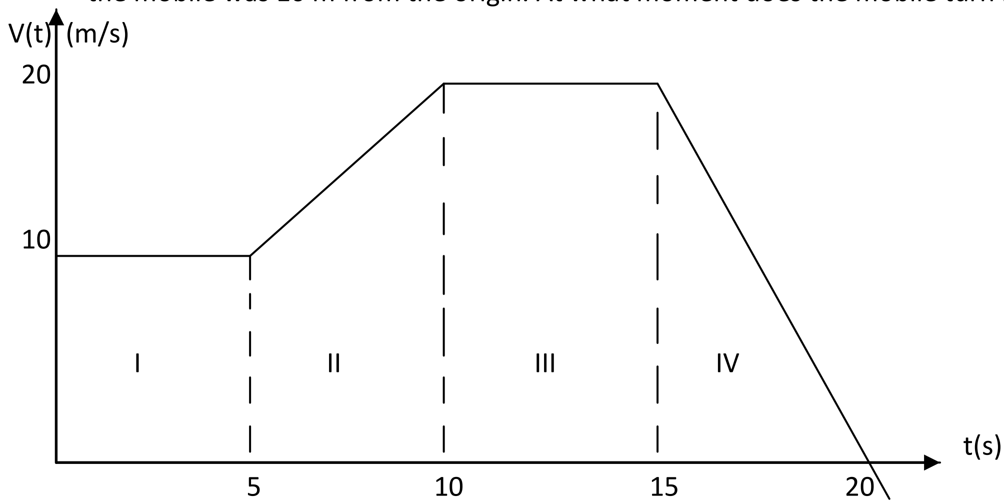
$x(t) = x(t + \frac{2\pi}{\omega})$ of period $T = \frac{2\pi}{\omega}$ and frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$.

Its velocity $V(t) = -A\omega sin(\theta_0 + \omega t)$

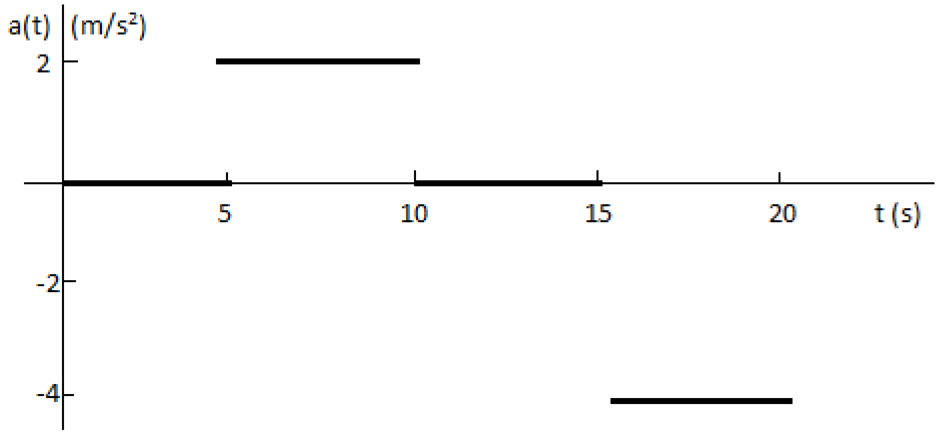
and its acceleration $a(t) = -A\omega^2 cos(\theta_0 + \omega t) = -\omega^2 x(t)$

Exercises :

1. The diagram of the velocities of a mobile A animated by a rectilinear movement on an axis OX is given by the figure below:
    - Plot the diagram of acceleration as a function of time.
    - What are the different phases of movement and their nature.
    - Determine the position of the mobile at times t = 10 s, and t = 20 s, knowing that initially the mobile was 10 m from the origin. At what moment does the mobile turn back?



The acceleration graph :



- Between 0 and 5 s, the speed is constant → the acceleration $a_1 = 0$ → Uniform rectilinear motion.
- Between 5 s and 10 s, the speed varies uniformly → the acceleration is represented by the slope $a_2 = \frac{10}{5} = 2\ m/s^2$ → Rectilinear uniformly varied motion.
- Between 10 s and 15 s, the speed is constant → the acceleration $a_3 = 0$ → Uniform rectilinear motion.
- Between 15 s and 20 s, the speed varies uniformly → the acceleration is represented by the slope $a_4 = \frac{-20}{5} = -4\ m/s^2$ → Rectilinear uniformly varied motion.
- The displacement of the mobile at t = 10 s is represented by the initial distance and the surfaces S_I and S_{II} in the graph V(t) → X(10) = 10 + 5x10 + (10 +20)x5/2 = 135 m

- The displacement of the mobile at t = 20 s is represented by the initial distance and the sum of the areas S_I , S_{II} , S_{III} and S_{IV} in the graph V(t) :
  → X(20) = 10 + 5*10 + (10 +20)*5/2 + 5*20 + 20*5/2 = 285 m
- At instant t = 20 s, the speed of the mobile changes sign and becomes negative
  → the mobile turns back at this instant.

2. An object is thrown up from the top of a building with a speed of 12.0 m/s. It reaches the ground 4.25 s later.
   - What is the maximum height reached by the object ?
   - How tall is the building ?
   - How fast does it hit the ground ?
   - The movement of the object is uniformly varied. It reaches its maximum height when its speed vanishes. We choose the *OZ* axis oriented positively upwards, with origin the top of the building. The maximum height reached by the object is therefore :
     $H_{max} = z(t) = \frac{V(t)^2 - V(0)^2}{-2g} = \frac{V(0)^2}{2g} \approx 7.35\ m$
   - The height of the building is equal to the absolute value of the abscissa of the object at the time of its collision with the ground (i.e. at t = 4.25 s) : $z(t) = V(0)t - \frac{1}{2}\ gt^2$
     $h = |z(4.25)| \approx 37.5\ m$
   - The speed of the collision of the object with the ground (i.e. at t = 4.25 s) :
     $V(t) = V(0) - gt$
     $V = V(4.25) \approx -29.65\ m/s$; the sign (–) results from the orientation of the axis.

3. A mobile moves along a straight line with the acceleration : $a = 9 - t^2$.
   - Find the expressions for velocity and displacement as a function of time in considering the following conditions : $t = 3\ s;\ v = 2\ m/s$ and $x = 7\ m$.
- By integrating the expression of the acceleration we obtain the hourly equation of the velocity : $v(t) = v(3) + \int_3^t a(\tau)d\tau = 2 + \int_3^t (9 - \tau^2)d\tau$
  Therefore, $v(t) = -\frac{1}{3}t^2 + 9t - 16$
  - By integrating the expression of the velocity, we obtain the time equation of the position : $x(t) = x(3) + \int_3^t v(\tau)d\tau = 2 + \int_3^t (-\frac{1}{3}\tau^2 + 9\tau - 16)d\tau$
    Therefore, $x(t) = -\frac{1}{9}t^3 + \frac{9}{2}t^2 - 16t + \frac{25}{2}$

4. The rectilinear movement of a mobile is defined by the equation : $s = 3t^3 - 2t^2 + 12t + 1$.
   - Calculate velocity and acceleration
   - Investigate the nature of motion
   - The mobile velocity is $v = \frac{ds}{dt} = 9t^2 - 4t + 12$
     The velocity v is always positive since $\Delta = -416 < 0$ and the coefficient of t^2> 0
   - The acceleration of the mobile is $a = \frac{dv}{dt} = 18t - 4$

- The movement is accelerated or retarded according to the sign of the product "$a \cdot v$". As for the direction of the movement, it is indicated by the sign of v which is positive. So the movement is accelerated for the values of $t > 2/9$ and decelerated for $0 < t < \frac{2}{9}$.

5. A mobile moves along the OX axis with a velocity $v = 2t - 5$ (m/s)
   - Calculate its acceleration as well as its time equation knowing that at the initial moment $t = 0$, the mobile was at the point of abscissa $x = 6$ (m).
   - Investigate the nature of motion
   - The acceleration of the mobile is $a = \frac{dv}{dt} = 2$ (m/s ^{2}), the acceleration is constant.
   - By integrating the expression of the velocity, we obtain the time equation :
     $x(t) = x(0) + \int_0^t v(\tau)d\tau = 6 + \int_0^t (2\tau - 5)d\tau = t^2 - 5t + 6$
   - Let us study the sign of "$a * v$" $= 2(2t - 5)$: it is positive for$t > 5/2$ (accelerated movement) and negative for$t < 5/2$ (decelerated movement).

6. A mobile moves in rectilinear motion. Its acceleration is given by $a = -\frac{\pi^2}{4}\, x$
   knowing that at the moment $t = 1$ s, the mobile was at the point of abscissa $x = 1\, m$ with a velocity $v = -\frac{\pi}{2\sqrt{3}}\, m/s$
   - Determine the nature of the movement, write its time equation.
   - We notice that we have a differential equation which is the characteristic equation of sinusoidal rectilinear motion : $a + \omega^2\, x = 0$ with $\omega^2 = \frac{\pi^2}{4}$
     and its solution is of the form : $x(t) = Acos(\omega t + \varphi)$ with $x(1) = 1$
     with a velocity $v(t) = -A\omega sin(\omega t + \varphi)$ with $v(1) = -\frac{\pi}{2\sqrt{3}}$
     $x(1) = Acos(\omega + \varphi) = 1$ and $v(1) = -A\omega sin(\omega + \varphi) = -\frac{\pi}{2\sqrt{3}}$
     Therefore : $\omega tan(\omega + \varphi) = \frac{\pi}{2\sqrt{3}}$
     Since $\omega = \frac{\pi}{2}$ then : $tan(\omega + \varphi) = \frac{1}{\sqrt{3}} \quad \rightarrow \quad \omega + \varphi = \frac{\pi}{6} \;\rightarrow\; \varphi = -\frac{\pi}{3}$
     $\rightarrow A = \frac{1}{cos(\omega+\varphi)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$
     Therefore, the time equation of rectilinear sinusoidal motion is $x(t) = \frac{2}{\sqrt{3}} cos(\frac{\pi}{2}t - \frac{\pi}{3})$
     Pulse $\omega = \frac{\pi}{2}\, rad/s$, amplitude $A = \frac{2}{\sqrt{3}}\, m$ and initial phase $\varphi = -\frac{\pi}{3}\, rad.$

7. The plane motion of a mobile is defined by : $x = sin^2(t)$ and $y = 1 + \cos{(2t)}$
   - Determine the trajectory of movement.
   - Calculate the coordinates of the velocity vector and those of the acceleration vector.
   - Knowing that $\cos(2t) = 1 + 2sin^2(t)$, we obtain the following relationship between x and y : $y = 2(1 - x)$. This is a straight line. Since $0 \leq x \leq 1$ and $0 \leq y \leq 2$:
     The trajectory is therefore a line segment joining the points (1,0) and (0,2).
   - The velocity vector has the coordinates :

$v_x = 2\sin(t)\cos(t) = \sin(2t)$ and $v_y = -2\sin(2t)$

Therefore : $v_y = -2v_x$ with $v_x(0) = v_y(0) = 0$ and $v_x\left(\frac{\pi}{2}\right) = v_y\left(\frac{\pi}{2}\right) = 0$

- acceleration vector has the coordinates : $a_x = 2\cos(2t)$ and $a_y = -\cos(2t)$

  $a_y = -\frac{1}{2}a_x$ with $a_x(0) = 2$, $a_y(0) = -1$ and $a_x\left(\frac{\pi}{2}\right) = -2$, $a_y\left(\frac{\pi}{2}\right) = 1$

8. A particle moves in an *XY plane* according to the law : $a_x = 2sint$ and $a_y = 3cost$.
   Knowing that for *t* = 0, we have: $x = 0$, $y = -2$ and $V_x = -2$, $V_y = 0$.
   - Find the equation of motion.
   - Find the equation of the trajectory.

- $v_x(t) - v_x(0) = \int_0^t a_x d\tau \quad \rightarrow \quad v_x(t) = -2cost$

  $v_y(t) - v_y(0) = \int_0^t V_y d\tau \quad \rightarrow \quad v_y(t) = 3sint$

  $x(t) - x(0) = \int_0^t v_x d\tau \quad \rightarrow \quad x(t) = -2sint$

  $y(t) - y(0) = \int_0^t v_y d\tau \quad \rightarrow \quad y(t) = -3cost + 1$
- By using the trigonometric property : $(sint)^2 + (cost)^2 = 1$

  $\rightarrow \frac{x^2}{4} + \frac{(y-1)^2}{9} = 1$

  The trajectory is an ellipse centered at (0,1) with minor axis a = 2 and major axis b = 3.

9. Consider a mobile M in motion such that : $\overrightarrow{\boldsymbol{OM}} = 6cost\vec{\imath} + 6sint\vec{\jmath} + (8t-3)\vec{k}$
   - Determine the nature of the trajectory of M in space (O, X, Y, Z) ?
   - Express $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ in cylindrical coordinates and determine their modulus ?
   - Find $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ in *Frenet 's* frame ?
   - Deduce the radius of curvature ρ ?

- The nature of the trajectory in the plane (O, X, Y) :

  $x = 6cost$ and $y = 6sint$

  Therefore : $x^2 + y^2 = 36$

  The movement in the plane (O, X, Y) is circular with radius R = 6 m and center (0, 0)

  Along the OZ axis : $z = 8t - 3$; the movement is rectilinear along OZ.

  Therefore, the motion along space (O, X, Y, Z) is helical.
- The position of the mobile M, its speed and its acceleration are therefore expressed in cylindrical coordinates as :

  $\overrightarrow{\boldsymbol{OM}} = 6\vec{\mathbf{u}}_\rho + (8t-4)\vec{\boldsymbol{k}}$

  $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{\boldsymbol{OM}}}{dt} = 6\vec{\mathbf{u}}_\theta + 8\vec{\boldsymbol{k}}$ with $|\vec{\boldsymbol{v}}| = 10\ m/s$

  $\vec{\boldsymbol{a}} = \frac{d\vec{v}}{dt} = -6\vec{\mathbf{u}}_\rho$ with $|\vec{\boldsymbol{a}}| = 6\ m/s^2$
- $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ in the *Frenet* frame *:*

  $\vec{\boldsymbol{v}} = 10\vec{\mathbf{u}}_T$

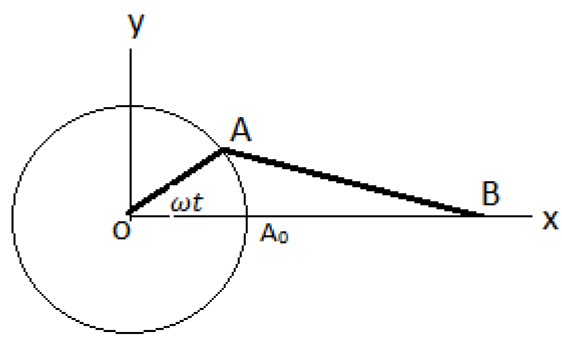
  $\vec{\boldsymbol{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_N = a_T\vec{\mathbf{u}}_T + a_N\vec{\mathbf{u}}_N = \frac{dv}{dt}\vec{\mathbf{u}}_T + \frac{v^2}{\rho}\vec{\mathbf{u}}_N$

Since $a_T = \frac{dv}{dt} = 0$, therefore $\vec{\boldsymbol{a}} = \frac{v^2}{\rho}\vec{\mathbf{u}}_N = 6\vec{\mathbf{u}}_N$

$\frac{v^2}{\rho} = a_N \quad \rightarrow \quad \rho = \frac{v^2}{a_N}$

Therefore, the radius of curvature is : $\rho = \frac{50}{3}\ m$

10. An arm *OA* rotating with a uniform angular speed ω around an axis *OZ*, is articulated at *A* with a rod *AB* . The rod *AB* is attached to a slider *B* which can slide along the axis *OX* . The arm and rod may cross as the rod passes behind the articulation *O*.
Knowing that *AB* = *L* and *OA* = *R* :
    - Find the equation of motion of *B* , knowing that B passes through A_0 at time *t* = 0,
    - At what instants does the velocity vanish?



- Making use of the cosine law : $AB^2 = OA^2 + OB^2 - 2OA.OBcos(\omega t)$
$L^2 = R^2 + x^2 - 2xRcos(\omega t) \ \rightarrow \ x(t) = Rcos(\omega t) + \sqrt{L^2 - R^2\sin^2(\omega t)}$
It can be checked that $x(0) = R + L$
- $v(t) = \frac{dx}{dt} = -\omega Rsin(\omega t) - \frac{\omega R^2 \sin(2\omega t)}{2\sqrt{L^2 - R^2\sin^2(\omega t)}}$
- The speed vanishes when $\omega t = k\pi \quad \rightarrow \quad t = \frac{k\pi}{\omega}$ where k is an integer.

11. Consider the position vector of a mobile $\vec{\boldsymbol{r}} = 2t^2\vec{\imath} + (5 - 3t)\vec{\jmath} - t^3\vec{k}$.
    - Calculate its velocity and acceleration.
    - Study the nature of motion.

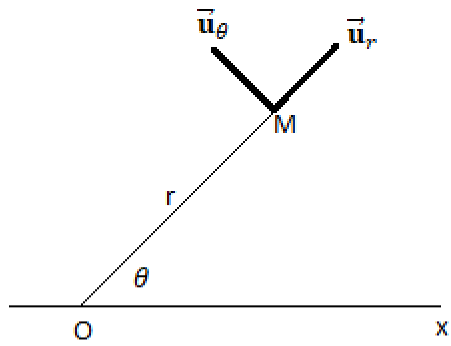
- Mobile velocity $\vec{\boldsymbol{v}} = \frac{d\vec{r}}{dt} = 4t\vec{\imath} - 3\vec{\jmath} - 3t^2\vec{k}$
Its modulus $|\vec{\boldsymbol{v}}| = \sqrt{9 + 16t^2 + 9t^4}$
- Mobile acceleration $\vec{\boldsymbol{a}} = \frac{d\vec{v}}{dt} = 4\vec{\imath} - 6t\vec{k}$
Its modulus $|\vec{\boldsymbol{a}}| = \sqrt{16 + 36t^2}$
The scalar product $\vec{\boldsymbol{a}}.\vec{\boldsymbol{v}} = 16t + 18t^3$ is positive
Consequently, the movement of the mobile is accelerated.
- The radius of curvature is given by the relation : $\rho = \frac{v^3}{|\vec{v}\wedge\vec{a}|}$
$\vec{\boldsymbol{v}}\wedge\vec{\boldsymbol{a}} = 18t\vec{\imath} + 12t^2\vec{\jmath} + 12\vec{k}$

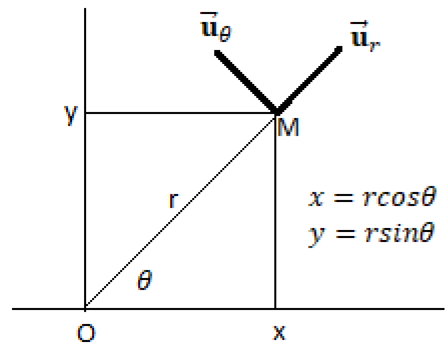
Therefore, the radius of curvature : $\rho = \frac{(9+16t^2+9t^4)^{3/2}}{(144+324t^2+144t^4)^{1/2}}$

12. In cylindrical coordinates, the position vector $\overrightarrow{\boldsymbol{OM}} = (t^2+2)\vec{\mathbf{u}}_\rho + 2t\vec{\boldsymbol{k}}$ with $\theta = 2t$
    - Calculate its speed and acceleration in the base $(\vec{\mathbf{u}}_{\rho,}\vec{\mathbf{u}}_\theta, \vec{\boldsymbol{k}})$
    - Calculate its speed and acceleration in the base $(\vec{\imath}, \vec{\jmath}, \vec{k})$
    - The velocity vector is $\vec{\boldsymbol{V}} = \dot{\rho}\vec{\mathbf{u}}_\rho + \rho\dot{\theta}\vec{\mathbf{u}}_\theta + \dot{z}\vec{\boldsymbol{k}} = 2t\vec{\mathbf{u}}_\rho + 2(t^2+2)\vec{\mathbf{u}}_\theta + 2\vec{\boldsymbol{k}}$
    - The acceleration vector is $\vec{\boldsymbol{a}} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{\mathbf{u}}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{\mathbf{u}}_\theta + \ddot{z}\vec{\boldsymbol{k}}$
      $\vec{\boldsymbol{a}} = -(4t^2+6)\vec{\mathbf{u}}_\rho + 8t\vec{\mathbf{u}}_\theta$
    - Because $\vec{\mathbf{u}}_\rho = \vec{\boldsymbol{\imath}}cos\theta + \vec{\boldsymbol{\jmath}}sin\theta$ and $\vec{\mathbf{u}}_\theta = -\vec{\boldsymbol{\imath}}sin\theta + \vec{\boldsymbol{\jmath}}cos\theta \rightarrow$
      $\vec{\boldsymbol{V}} = 2\,[tcos(2t) - (t^2+2)\sin(2t)]\vec{\boldsymbol{\imath}} + 2\,[tsin(2t) + (t^2+2)\cos(2t)]\vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}}$
      $\vec{\boldsymbol{a}} = -[(4t^2+6)\cos(2\mathrm{t}) + 8\mathrm{t}\sin(2\mathrm{t})]\vec{\boldsymbol{\imath}} - [(4t^2+6)\sin(2\mathrm{t}) - 8\mathrm{t}\cos(2\mathrm{t})]\vec{\boldsymbol{\jmath}}$

13. We give the parametric equations of the plane trajectory of a moving point with respect to a Cartesian frame of reference : $x = 3t$ and $y = 2t^2 - 3t$
    - Determine the equation of the trajectory, What is its shape ?
    - Calculate the speed of the mobile
    - Show that its acceleration is constant
    - Determine the normal and tangential components of the acceleration in a *Frenet's* frame and deduce the radius of curvature.
    - $x = 3t \rightarrow t = \frac{x}{3} \rightarrow y = \frac{2}{9}\,x^2 - x$
      $\rightarrow$ the trajectory is a parabola.
    - The speed of the mobile : $v_x(t) = \frac{dx}{dt} = 3$ and $v_y(t) = \frac{dx}{dt} = 4t - 3$
      The speed module : $|\vec{\boldsymbol{v}}| = \sqrt{16t^2 - 24t + 18}\,m/s$
    - The acceleration of the mobile : $a_x(t) = \frac{dv_x}{dt} = 0$ and $a_y(t) = \frac{dv_y}{dt} = 4$
      The acceleration of the mobile is constant : $|\vec{\boldsymbol{a}}| = 4\,m/s^2$
    - In *Frenet 's* frame , the tangential acceleration : $a_T(t) = \frac{d|\vec{\boldsymbol{v}}|}{dt} = \frac{16t-12}{\sqrt{16t^2-24t+18}}$
      The normal acceleration : $a_N(t) = \sqrt{a^2 - {a_T}^2} = \frac{12}{\sqrt{16t^2-24t+18}}$
    - The radius of curvature : $\rho = \frac{v^2}{a_N} = \frac{(16t^2-24t+18)^{3/2}}{12}\,m$

14. Convert the Vector $\vec{\boldsymbol{A}} = A_x\vec{\imath} + A_y\vec{\jmath}$
    to polar coordinates from Cartesian coordinates :
    - We have : $\vec{\mathbf{u}}_r = \vec{\imath}cos\theta + \vec{\jmath}sin\theta$ and $\vec{\mathbf{u}}_\theta = -\vec{\imath}sin\theta + \vec{\jmath}cos\theta$
      $\rightarrow \vec{\imath} = \vec{\mathbf{u}}_r cos\theta - \vec{\mathbf{u}}_\theta sin\theta$ and $\vec{\jmath} = \vec{\mathbf{u}}_r sin\theta + \vec{\mathbf{u}}_\theta cos\theta$





$\vec{A} = A_x(\vec{\mathbf{u}}_r cos\theta - \vec{\mathbf{u}}_\theta sin\theta) + A_y(\vec{\mathbf{u}}_r sin\theta + \vec{\mathbf{u}}_\theta cos\theta)$

$\vec{A} = (A_x cos\theta + A_y sin\theta)\vec{\mathbf{u}}_r + (A_y cos\theta - A_x sin\theta)\vec{\mathbf{u}}_\theta = A_r\vec{\mathbf{u}}_r + A_\theta\vec{\mathbf{u}}_\theta$

Therefore,

$$\begin{cases} A_r = A_x cos\theta + A_y sin\theta \\ A_\theta = A_y cos\theta - A_x sin\theta \end{cases}$$

15. Convert the Vector $\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$
    to spherical coordinates from Cartesian coordinates.

- We have :

$$\begin{cases} \vec{\mathbf{u}}_r = \vec{\imath}cos\varphi sin\theta + \vec{\jmath}sin\varphi sin\theta + \vec{k}cos\theta \\ \vec{\mathbf{u}}_\varphi = -\vec{\imath}sin\varphi + \vec{\jmath}cos\varphi \\ \vec{\mathbf{u}}_\theta = \vec{\imath}cos\varphi cos\theta + \vec{\jmath}sin\varphi cos\theta - \vec{k}sin\theta \end{cases}$$

$\vec{A} = A_r\vec{\mathbf{u}}_r + A_\varphi\vec{\mathbf{u}}_\varphi + A_\theta\vec{\mathbf{u}}_\theta$

$$\begin{cases} A_x = A_r cos\varphi sin\theta - A_\varphi sin\varphi + A_\theta cos\varphi cos\theta \\ A_y = A_r sin\varphi sin\theta + A_\varphi cos\varphi + A_\theta sin\varphi cos\theta \\ A_z = A_r cos\theta - A_\theta sin\theta \end{cases}$$

By inverting the coordinates, we get :

$$\begin{cases} A_r = A_x cos\varphi sin\theta + A_y sin\varphi sin\theta + A_z cos\theta \\ A_\theta = A_x cos\varphi cos\theta + A_y sin\varphi cos\theta - A_z sin\theta \\ A_\varphi = -A_x sin\varphi + A_y cos\varphi \end{cases}$$

<u>**Chapter 3**</u>

<u>**The relative movement**</u>

Let us consider a mobile M and the following two Cartesian coordinate systems: $\Re$ (O, X, Y, Z), assumed to be fixed, which is called absolute coordinate system and $\Re$ ' (O', X', Y', Z'), in any motion with respect to $\Re$ (O, X, Y, Z), which is the relative coordinate system.

3.1 Absolute movement:

The movement of the mobile M considered with respect to the absolute reference $\Re$ (O, X, Y, Z) is characterized by the quantities :

- the position vector $\overrightarrow{\boldsymbol{OM}} = \vec{\boldsymbol{r}} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$
- the absolute velocity $\vec{\mathbf{V}}_a = \frac{d\vec{r}}{dt} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$
- the absolute acceleration $\vec{\mathbf{a}}_a = \frac{d\vec{V}}{dt} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$

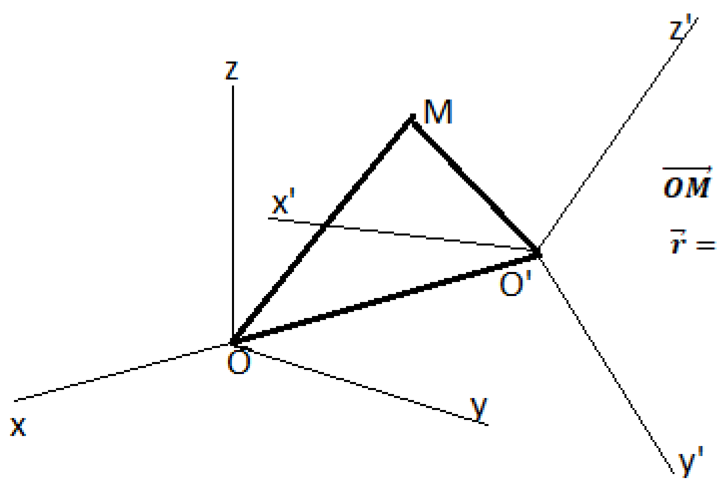
derivations are performed in $\Re$ in which the base$(\vec{\imath}, \vec{\jmath}, \vec{k})$ is invariable.

3.2 Relative motion:

The movement of the mobile M considered with respect to the reference $\Re$ ' (O', X', Y', Z'), is characterized by the quantities :

- the position vector $\overrightarrow{\boldsymbol{O'M}} = (\vec{\boldsymbol{r}}'|\Re') = x'\vec{\imath}' + y'\vec{\jmath}' + z'\vec{k}'$
- the relative velocity $\vec{\mathbf{V}}_r = (\frac{d\vec{r}'}{dt}\Big|\Re') = \dot{x}'\vec{\imath}' + \dot{y}'\vec{\jmath}' + \dot{z}'\vec{k}'$
- the relative acceleration $\vec{\mathbf{a}}_r = (\frac{d\vec{\mathbf{V}}_r}{dt}\Big|\Re') = \ddot{x}'\vec{\imath}' + \ddot{y}'\vec{\jmath}' + \ddot{z}'\vec{k}'$

the derivations are made in $\Re$' in which the base$(\vec{\imath}', \vec{\jmath}', \vec{k}')$ is invariable.



$$\overrightarrow{\boldsymbol{OM}} = \overrightarrow{\boldsymbol{OO'}} + \overrightarrow{\boldsymbol{O'M}}$$

$$\vec{\boldsymbol{r}} = \vec{\mathbf{r}}_0 + \vec{\boldsymbol{r}}'$$

3.3 Composition of the velocity vectors:

The Chasles relation allows us to write : $\overrightarrow{\boldsymbol{OM}} = \overrightarrow{\boldsymbol{OO'}} + \overrightarrow{\boldsymbol{O'M}}$

$$\overrightarrow{\boldsymbol{OM}} = \vec{\mathbf{r}}_0 + \vec{\boldsymbol{r}}' = \vec{\mathbf{r}}_0 + x'\vec{\imath}' + y'\vec{\jmath}' + z'\vec{\boldsymbol{k}}'$$

So the absolute velocity $\vec{\mathbf{V}}_a = \frac{d\overrightarrow{\boldsymbol{OM}}}{dt} = \frac{d\overrightarrow{\boldsymbol{OO'}}}{dt} + \frac{d\overrightarrow{\boldsymbol{O'M}}}{dt} = \frac{d\vec{\mathbf{r}}_0}{dt} + \frac{d\vec{\mathbf{r}}'}{dt}$

$$\vec{\mathbf{V}}_a = \left(\frac{d\vec{\mathbf{r}}_0}{dt} + x'\frac{d\vec{\boldsymbol{\imath}}'}{dt} + y'\frac{d\vec{\boldsymbol{\jmath}}'}{dt} + z'\frac{d\vec{\boldsymbol{k}}'}{dt}\right) + \dot{x}'\vec{\imath}' + \dot{y}'\vec{\jmath}' + \dot{z}'\vec{\boldsymbol{k}}'$$

$$\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r$$

with training velocity $\vec{\mathbf{V}}_e = \frac{d\vec{\mathbf{r}}_0}{dt} + x'\frac{d\vec{\boldsymbol{\imath}}'}{dt} + y'\frac{d\vec{\boldsymbol{\jmath}}'}{dt} + z'\frac{d\vec{\boldsymbol{k}}'}{dt}$ which represents the velocity of the reference $\mathfrak{R}'$ with respect to the reference $\mathfrak{R}$. The first term $\frac{d\vec{\mathbf{r}}_0}{dt}$ represents the translational velocity of the origin O' with respect to $\mathfrak{R}$ and the second term translates the change of orientation of the mobile frame $\mathfrak{R}'$ and relative speed $\vec{\mathbf{V}}_r = \dot{x}'\vec{\imath}' + \dot{y}'\vec{\jmath}' + \dot{z}'\vec{\boldsymbol{k}}'$

Let be $\vec{\boldsymbol{\Omega}}$ the vector representing the angular velocity of the change of orientation of the mobile frame $\mathfrak{R}'$ with respect to the fixed frame $\mathfrak{R}$. The training and relative velocities $\vec{\mathbf{V}}_e$ and $\vec{\mathbf{V}}_r$ are expressed by :

$$\vec{\mathbf{V}}_e = (\vec{\boldsymbol{V}}(O')|\mathfrak{R}) + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{r}}'|\mathfrak{R}')$$

$$\vec{\mathbf{V}}_r = \left(\frac{d\vec{\boldsymbol{r}}'}{dt}\right|\mathfrak{R}')$$

3.4 Composition of the vectors accelerations:

The absolute acceleration defined in the reference $\mathfrak{R}$ is :

$$\vec{\mathbf{a}}_a = \frac{d\vec{\mathbf{V}}_a}{dt}$$

$$\vec{\mathbf{a}}_a = \left(\frac{d^2\vec{\mathbf{r}}_0}{dt^2} + x'\frac{d^2\vec{\imath}'}{dt^2} + y'\frac{d^2\vec{\jmath}'}{dt^2} + z'\frac{d^2\vec{\boldsymbol{k}}'}{dt^2}\right) + (\ddot{x}'\vec{\imath}' + \ddot{y}'\vec{\jmath}' + \ddot{z}'\vec{k}') + 2\left(\dot{x}'\frac{d\vec{\boldsymbol{\imath}}'}{dt} + \dot{y}'\frac{d\vec{\boldsymbol{\jmath}}'}{dt} + \dot{z}\frac{d\vec{\boldsymbol{k}}'}{dt}\right)$$

$$\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_e + \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_c$$

with $\vec{\mathbf{a}}_e = \left(\frac{d^2\vec{\mathbf{r}}_0}{dt^2} + x'\frac{d^2\vec{\imath}'}{dt^2} + y'\frac{d^2\vec{\jmath}'}{dt^2} + z'\frac{d^2\vec{\boldsymbol{k}}'}{dt^2}\right)$ representing the training acceleration of the point coinciding with respect to the absolute reference $\mathfrak{R}$.

$\vec{\mathbf{a}}_r = (\ddot{x}'\vec{\imath}' + \ddot{y}'\vec{\jmath}' + \ddot{z}'\vec{k}')$ representing the relative acceleration, and

$\vec{\mathbf{a}}_c = 2\left(\dot{x}'\frac{d\vec{\imath}'}{dt} + \dot{y}'\frac{d\vec{\boldsymbol{\jmath}}'}{dt} + \dot{z}'\frac{d\vec{\boldsymbol{k}}'}{dt}\right)$ is a complementary acceleration, called Coriolis acceleration.

These accelerations can be expressed as a function of the vector $\vec{\boldsymbol{\Omega}}$ representing the angular velocity of the change in orientation of the mobile frame $\mathfrak{R}'$ relative to the fixed frame $\mathfrak{R}$ by :

$$\vec{\mathbf{a}}_e = (\vec{\boldsymbol{a}}(O')|\mathfrak{R}) + (\dot{\vec{\boldsymbol{\Omega}}}\wedge\vec{\boldsymbol{r}}'|\mathfrak{R}') + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{\Omega}}\wedge\vec{\boldsymbol{r}}'|\mathfrak{R}')$$

$$\vec{\mathbf{a}}_r = \left(\frac{d\vec{\mathbf{V}}_r}{dt}\right|\mathfrak{R}')$$

$$\vec{\mathbf{a}}_c = 2\vec{\boldsymbol{\Omega}}\wedge(\vec{\mathbf{V}}_r|\mathfrak{R}')$$

It can be concluded that if two frames are in uniform rectilinear translation with respect to each other, then the accelerations of a material point M measured in one or the other of these frames are equal; this type of frames are called "Galilean frames".

We can also conclude that if the moving frame $\mathfrak{R}'$is in translation with respect to the fixed frame $\mathfrak{R}$ or if the moving body M is fixed with respect to the moving frame $\mathfrak{R}'$, the Coriolis acceleration is zero.

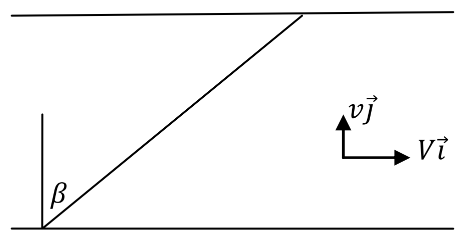
Exercises :

1. A swimmer crosses a river, of width L = 0.25 km, from one bank to the other, perpendicular to the current, at a constant speed v = 2 $km\,/\,h$. The speed of the current is V = 1 $km\,/\,h$. These speeds are measured in relation to an observer placed on one of the banks.
   - What is the trajectory, the speed of the swimmer and the time taken to reach the other shore.
   - What should be his direction of departure if the swimmer wants to reach a point directly opposite to the other bank. Deduce the time taken to reach the opposite bank in this case.

- The swimmer is deflected by an angle β from his initial direction such that : $tan\beta = \frac{V}{v} = \frac{1}{2}$

$$\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r = V\vec{\imath} + v\vec{\jmath}$$

The absolute velocity module $|\vec{\mathbf{V}}_a| = \sqrt{V^2 + v^2}$



- The position of the swimmer is :

$$x = Vt \quad \text{and} \quad y = vt$$

- Its trajectory is a straight line : $y = \frac{v}{V}x$
- The time taken by the swimmer to reach the other shore :

$$T = \frac{L}{v} = 0.125\,h$$

- The distance travelled :

$$D = |\vec{\mathbf{V}}_a|T = L\sqrt{1 + \frac{V^2}{v^2}} = 0.125\sqrt{5}\,km$$

- To reach a point directly opposite the other shore, the absolute velocity $\vec{\mathbf{V}}_a$ must be perpendicular to both shores with a magnitude :

$$|\vec{\mathbf{V}}_a| = \sqrt{v^2 - V^2}$$

The swimmer's speed must be oriented opposite the current with an angle α :

$$\alpha = arctan\frac{V}{\sqrt{v^2 - V^2}} = arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

- The time taken to reach the opposite bank $T' = \frac{L}{\sqrt{v^2 - V^2}} = \frac{0.25}{\sqrt{3}}\ h$

2. A mobile is described by the position vector in a fixed frame **R** by $\overrightarrow{\boldsymbol{OM}} = 3t\vec{\imath} + t^2\vec{\jmath} + (3t+4)\vec{k}$ and by $\overrightarrow{\boldsymbol{O'M}} = (2-3t)\vec{\imath}' + (t^2 - t)\vec{\jmath}' + t\vec{k}'$ in a mobile frame ***R'*** . The latter is in rectilinear motion with respect to ***R.***
   - Determine the absolute velocity and the relative velocity of M.
   - Deduce the driving speed and the nature of the motion of **R'** with respect to ***R***.
   - Determine the absolute acceleration and the relative acceleration of the mobile M.

- absolute velocity $\vec{\mathbf{V}}_a = \frac{d\overrightarrow{\boldsymbol{OM}}}{dt} = 3\vec{\imath} + 2t\vec{\jmath} + 3\vec{k}$
- relative velocity $\vec{\mathbf{V}}_r = \frac{d\overrightarrow{\boldsymbol{O'M}}}{dt} = -3\vec{\imath}' + (2t-1)\vec{\jmath}' + \vec{k}'$

Since the two frames are in rectilinear motion from each other :

$$\vec{\imath} = \vec{\imath}', \quad \vec{\jmath} = \vec{\jmath}' \quad \text{and} \quad \vec{k} = \vec{k}'$$

- Using the velocity decomposition relation : $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r$

Therefore the training velocity is : $\vec{\mathbf{V}}_e = \vec{\mathbf{V}}_a - \vec{\mathbf{V}}_r = 6\vec{\imath} + \vec{\jmath} + 2\vec{k}$

Its modulus $|\vec{\mathbf{V}}_e| = \sqrt{41}$ is constant and independent of time t.

So the motion of **R'** with respect to ***R*** is a uniform rectilinear motion.

- The absolute acceleration of mobile M : $\vec{\mathbf{a}}_a = \frac{d\vec{\mathbf{V}}_a}{dt} = 2\vec{\jmath}$
- The relative acceleration of mobile M : $\vec{\mathbf{a}}_r = (\frac{d\vec{\mathbf{V}}_r}{dt}\Big| \mathrm{R}') = 2\vec{\jmath}$

  $\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_r$ because the motion of ***R'*** with respect to R is a uniform rectilinear motion.

3. A mobile is described by the position vector in a mobile frame ***R'*** by :
   $\overrightarrow{\boldsymbol{O'M}} = 5t\vec{\imath}' + (2t^2 - t)\vec{\jmath}' - 2t\vec{k}'$. This reference is in rectilinear translation movement of velocity vector $\vec{\mathbf{V}}_e = 2t\vec{\imath} + \vec{\jmath} + \vec{k}$ with respect to a fixed reference ***R.***
   - Find the expression of the absolute speed of M with respect to the frame ***R***.
   - Deduce the position vector of M in the fixed frame ***R***, knowing that at time t = 0 , M is at the point (0, 1, -2) in the frame ***R.***
   - Calculate the relative acceleration $\vec{\mathbf{a}}_r$ and absolute acceleration $\vec{\mathbf{a}}_a$ of the mobile M.

- Relative velocity $\vec{\mathbf{V}}_r = \frac{d\overrightarrow{\boldsymbol{O'M}}}{dt} = 5\vec{\imath}' + (4t - 1)\vec{\jmath}' - 2\vec{k}'$

  Since the two frames are in rectilinear motion from each other :

  $\vec{\imath} = \vec{\imath}'$, $\vec{\jmath} = \vec{\jmath}'$ and $\vec{k} = \vec{k}'$

  Using the velocity decomposition relation : $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r$

  Then the absolute speed $\vec{\mathbf{V}}_a(t) = (2t + 5)\vec{\imath} + 4t\vec{\jmath} - \vec{k}$
- the position vector of M in the fixed frame ***R*** :

  $\overrightarrow{\boldsymbol{OM}}(t) = \overrightarrow{\boldsymbol{OM}}(0) + \int_0^t \vec{\mathbf{V}}_a(\tau)d\tau = \vec{\jmath} - 2\vec{k} + (t^2 + 5t)\vec{\imath} + 2t^2\vec{\jmath} - t\vec{k}$

  So, $\overrightarrow{\boldsymbol{OM}}(t) = (t^2 + 5t)\vec{\imath} + (2t^2 + 1)\vec{\jmath} - (t + 2)\vec{k}$
- The relative acceleration is : $\vec{\mathbf{a}}_r = (\frac{d\vec{\mathbf{V}}_r}{dt}\Big| \mathrm{R}') = 4\vec{\jmath}$
- The absolute acceleration is : $\vec{\mathbf{a}}_a = \frac{d\vec{\mathbf{V}}_a}{dt} = 2\vec{\imath} + 4\vec{\jmath}$

  We can verify that for this case : $\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_e + \vec{\mathbf{a}}_r$ because $\vec{\mathbf{a}}_c = 0$

4. A moving frame **R'** (OX', OY', OZ) is rotating relative to a fixed frame **R** (OX , OY , OZ) following the OZ axis with a constant angular velocity Ω .
   A mobile M moves on the line OX' according to the law : $x' = A\cos\Omega t$, where A is a constant.
   - Determine the relative speed and the training velocity of M by their projections in the moving frame X'OY' at time t as a function of A and Ω.
   - Deduce the absolute velocity expressed in this same projection base.
   - Show that its modulus is constant.
   - Determine the relative acceleration, the training acceleration and the Coriolis acceleration of M by their projections into the moving frame X'OY' at time t as a function of A and Ω.
   - Deduce the absolute acceleration expressed in this same projection base, and show that its modulus is constant.

- $\overrightarrow{\boldsymbol{OM}}(t) = \mathrm{A}\cos\Omega t\vec{\imath}'$ and $\vec{\boldsymbol{\Omega}} = \Omega\vec{k}'$

  Knowing that : $\vec{\imath}' = cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}$, $\vec{\jmath}' = -sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}$ and $\vec{k}' = \vec{k}$

  The relative velocity $\vec{\mathbf{V}}_r = \left(\frac{d\overrightarrow{\boldsymbol{OM}}(t)}{dt}\middle| \mathrm{R}'\right) = -\mathrm{A}\Omega\sin\Omega t\vec{\imath}'$

  $\vec{\mathbf{V}}_r = -\mathrm{A}\Omega\sin\Omega t[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}\,]$

  The training velocity $\vec{\mathbf{V}}_e = \left(\vec{\boldsymbol{V}}(O')\middle|\mathrm{R}\right) + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{r}}'|\mathrm{R}') = 0 + \Omega\vec{k}'\wedge\mathrm{A}\cos\Omega t\,\vec{\imath}'$

  $\vec{\mathbf{V}}_e = A\Omega\cos\Omega t\vec{\jmath}' = A\Omega\cos\Omega t[-sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}]$

- absolute velocity $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r = A\Omega\cos\Omega t\vec{\jmath}' - \mathrm{A}\Omega\sin\Omega t\vec{\imath}' = A\Omega[-\sin\Omega t\vec{\imath}' + \cos\Omega t\vec{\jmath}']$

  It can be verified that : $\vec{\mathbf{V}}_a = A\Omega[-\sin 2\Omega t\vec{\imath} + \cos 2\Omega t\vec{\jmath}]$

  $\left|\vec{\mathbf{V}}_a\right| = A\Omega$, the modulus of the absolute velocity is constant.

- The relative acceleration $\vec{\mathbf{a}}_r = (\frac{d\vec{\mathbf{V}}_r}{dt}\Big| \mathrm{R}') = -\mathrm{A}\Omega^2\cos\Omega t\vec{\imath}'$

  $\vec{\mathbf{a}}_r = -\mathrm{A}\Omega^2\cos\Omega t[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}]$

  The training acceleration $\vec{\mathbf{a}}_e = (\vec{\boldsymbol{a}}(O')|\mathrm{R}) + (\dot{\vec{\boldsymbol{\Omega}}}\wedge\vec{\boldsymbol{r}}'|\mathrm{R}') + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{\Omega}}\wedge\vec{\boldsymbol{r}}'|\mathrm{R}')$

  $\vec{\mathbf{a}}_e = 0 + 0 + \Omega\vec{k}'\wedge(\Omega\vec{k}'\wedge\mathrm{A}\cos\Omega t\,\vec{\imath}' = -A\Omega^2\cos\Omega t\vec{\imath}'$

  $\vec{\mathbf{a}}_e = -A\Omega^2\cos\Omega t[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}]$

  Coriolis acceleration $\vec{\mathbf{a}}_c = 2\vec{\boldsymbol{\Omega}}\wedge(\vec{\mathbf{V}}_r\middle|\mathrm{R}') = 2\Omega\vec{k}'\wedge(-\mathrm{A}\Omega\sin\Omega t\vec{\imath}') = -2A\Omega^2\sin\Omega t\vec{\jmath}'$

  $\vec{\mathbf{a}}_c = -2A\Omega^2\sin\Omega t[-sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}]$

  The absolute acceleration : $\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_e + \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_c$

  $\vec{\mathbf{a}}_a = -2\mathrm{A}\Omega^2\cos\Omega t\vec{\imath}' - 2A\Omega^2\sin\Omega t\vec{\jmath}'$

  $\vec{\mathbf{a}}_a = -2\mathrm{A}\Omega^2[cos2\Omega t\vec{\imath} + sin2\Omega t\vec{\jmath}]$

  We can verify that $\vec{\mathbf{a}}_a = \frac{d\vec{\mathbf{V}}_a}{dt}$

  $|\vec{\mathbf{a}}_a| = 2\mathrm{A}\Omega^2$, the modulus of absolute acceleration is constant.

5. A frame **R'** (OX', OY', OZ) rotating relative to a fixed **R** (OX , OY , OZ) along the OZ axis with a constant angular velocity Ω . A mobile M moves on the line OX' according to the law :
   $\overrightarrow{\boldsymbol{OM}}(t) = Ate^{\omega t}\vec{\imath}'$, where A is a constant.
   - Determine the relative, training and absolute velocities.
   - Determine the relative, training, Coriolis and absolute accelerations.

- relative velocity $\vec{\mathbf{V}}_r = \left(\frac{d\vec{r}'}{dt}\middle| \mathrm{R}'\right) = \mathrm{A}(1 + \omega t)e^{\omega t}\vec{\imath}'$

- training velocity $\vec{\mathbf{V}}_e = \left(\vec{\boldsymbol{V}}(O')\middle|\mathrm{R}\right) + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{r}}'|\mathrm{R}') = 0 + \Omega\vec{k}'\wedge Ate^{\omega t}\vec{\imath}' = A\Omega te^{\omega t}\vec{\jmath}'$,

  Knowing that : $\vec{\imath}' = cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}$, $\vec{\jmath}' = -sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}$ and $\vec{k}' = \vec{k}$

  absolute velocity $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r$

  $\vec{\mathbf{V}}_a =[-\Omega t sin\Omega t + (1 + \omega t)cos\Omega t]Ae^{\omega t}\vec{\imath} + [\Omega t cos\Omega t + (1 + \omega t)sin\Omega t]Ae^{\omega t}\vec{\jmath}$

- The relative acceleration $\vec{\mathbf{a}}_r = (\frac{d\vec{\mathbf{V}}_r}{dt}\Big| \mathrm{R}') = \mathrm{A}\omega[(2 + \omega t)]e^{\omega t}\vec{\imath}'$

  $\vec{\mathbf{a}}_r = \mathrm{A}\omega[(2 + \omega t)]e^{\omega t}[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}]$

  The training acceleration $\vec{\mathbf{a}}_e = (\vec{\boldsymbol{a}}(O')|\mathrm{R}) + (\dot{\vec{\boldsymbol{\Omega}}}\wedge\vec{\boldsymbol{r}}'|\mathrm{R}') + \vec{\boldsymbol{\Omega}}\wedge(\vec{\boldsymbol{\Omega}}\wedge\vec{\boldsymbol{r}}'|\mathrm{R}')$

  $\vec{\mathbf{a}}_e = 0 + 0 + \Omega\vec{k}'\wedge(\Omega\vec{k}'\wedge Ate^{\omega t}\vec{\imath}' = -A\Omega^2 te^{\omega t}\vec{\imath}' = -A\Omega^2 te^{\omega t}[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}]$

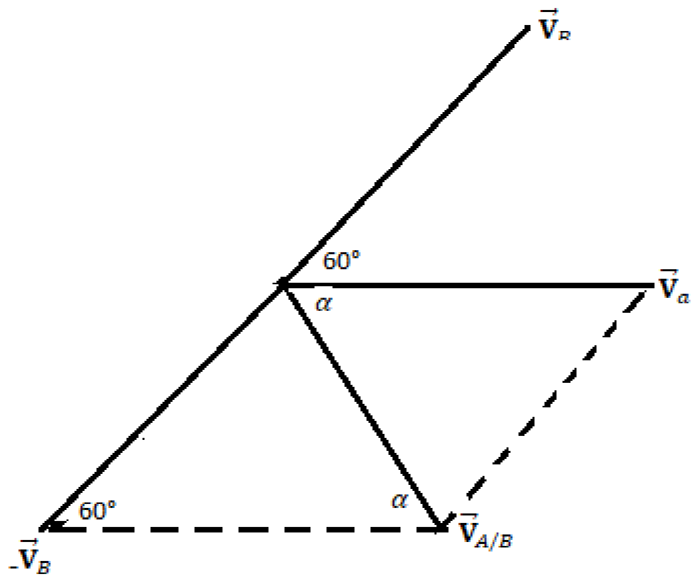
Coriolis acceleration $\vec{\mathbf{a}}_c = 2\vec{\boldsymbol{\Omega}}\wedge(\vec{\mathbf{V}}_r|\mathrm{R}') = 2\Omega\vec{k}'\wedge\mathrm{A}(1+\omega t)e^{\omega t}\vec{\imath}'$
$\vec{\mathbf{a}}_c = 2\Omega\mathrm{A}(1+\omega t)e^{\omega t}\vec{\jmath}' = 2\Omega\mathrm{A}e^{\omega t}(1+\omega t)[-sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}]$
The absolute acceleration : $\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_e + \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_c$
$\vec{\mathbf{a}}_a = Ae^{\omega t}\{[(\omega^2 - \Omega^2)t + 2\omega)][cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}] + [2\Omega(1+\omega t)][-sin\Omega t\vec{\imath} + cos\Omega t\vec{\jmath}]\}$
We can verify that $\vec{\mathbf{a}}_a = \frac{d\vec{\mathbf{V}}_a}{dt}$ with $\vec{\mathbf{V}}_a = (\frac{d\overrightarrow{\boldsymbol{OM}}(t)}{dt}\Big|\mathrm{R})$ and $\overrightarrow{\boldsymbol{OM}}(t) = Ate^{\omega t}[cos\Omega t\vec{\imath} + sin\Omega t\vec{\jmath}]$

6. A marble falls without initial velocity from a building of height H. Its fall is free according to a uniformly accelerated movement of acceleration g.
   - What is the trajectory of the ball in a reference frame linked to a car moving in a rectilinear and uniform movement of speed v and passing vertically when it is released ?
   - What is the trajectory of the ball in a frame of reference linked to a car moving in a uniformly accelerated rectilinear motion with acceleration $\boldsymbol{a}$ and passing to the falling vertical at the moment of release ?
   - The distance traveled by the ball in its free fall after a time t is : $z = H - \frac{1}{2}gt^2$
   The distance traveled by the car with a constant speed during the same time t is
   $x = vt$
   By elimination of time, we obtain the equation of the trajectory of the ball with respect to the moving frame linked to the car :
   $z = H - \frac{1}{2v^2}gx^2$; the trajectory is therefore a parabola.
   - The distance traveled by the ball in its free fall after a time t is : $z = H - \frac{1}{2}gt^2$
   The distance traveled by the car moving in a uniformly accelerated rectilinear motion of acceleration $\boldsymbol{a}$ during the same duration t is :
   $x = \frac{1}{2}\boldsymbol{a}t^2$
   By elimination of time, we obtain the equation of the trajectory of the ball with respect to the moving frame linked to the car :
   $z = H - \frac{g}{\boldsymbol{a}}x$; the trajectory is a straight line.

7. Two mobiles A and B move in two rectilinear paths with the respective speeds :
   V_A = 20 m/s and V_B = 30 m/s.
   - Determine the relative velocity vector of A with respect to B when the two mobiles are rolling in the same direction.
   - Determine the relative velocity vector of A with respect to B when the two mobiles are rolling in opposite directions.
   - Determine the relative velocity vector of A with respect to B when the two mobiles are now rolling on two paths which intersect forming an angle of 60° between them.
   - The relative speed of A compared to B : $\vec{\mathbf{V}}_{A/B} = \vec{\mathbf{V}}_A - \vec{\mathbf{V}}_B$
   $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ two unit vectors parallel to the two straight tracks.
   - The two mobiles roll in the same direction : $\vec{\mathbf{u}} = \vec{\mathbf{v}}$
   So, $\vec{\mathbf{V}}_{A/B} = 20\vec{\mathbf{u}} - 30\vec{\mathbf{u}} = \mathbf{-10}\vec{\mathbf{u}}$

- The two mobiles roll in opposite directions : $\vec{\mathbf{u}} = -\vec{\mathbf{v}}$

  So, $\vec{\mathbf{V}}_{A/B} = 20\vec{\mathbf{u}} + 30\vec{\mathbf{u}} = \mathbf{50}\vec{\mathbf{u}}$
- The two mobiles roll on two tracks which intersect forming an angle of 60° between them :



$$\vec{\mathbf{V}}_{A/B} = 20\vec{\mathbf{u}} - 30\vec{\mathbf{v}} \quad \rightarrow \quad |\vec{\mathbf{V}}_{A/B}| = \sqrt{20^2 + 30^2 - 2x20x30cos60} = 10\sqrt{7}\ m/s$$

The direction of the relative velocity vector $\vec{\mathbf{V}}_{A/B}$ is defined by the angle αsuch that :

$$\frac{|\vec{\mathbf{V}}_{A/B}|}{sin60} = \frac{|\vec{\mathbf{V}}_B|}{sin\alpha} \quad \rightarrow \quad sin\alpha = \frac{|\vec{\mathbf{V}}_B|}{|\vec{\mathbf{V}}_{A/B}|} sin60 = \frac{3}{2\sqrt{7}} \quad \rightarrow \quad \alpha = \arcsin\left(\frac{3\sqrt{3}}{2\sqrt{7}}\right)$$

$$\alpha \approx 79.11^0$$

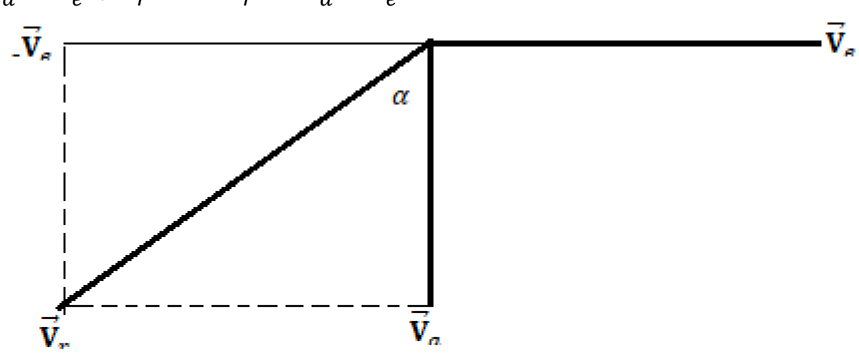
8. Snowflakes fall vertically with a speed of 30 m/s.
   - With what speed do these flakes hit the windshield of a car traveling with a constant speed of 40 m/s on a horizontal track.

- The speed of the car relative to the ground represents the training velocity.

  The speed of the snowflakes relative to the ground represents the absolute velocity.

  The speed of the snowflakes relative to the car represents the relative velocity.

  $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r \quad \rightarrow \quad \vec{\mathbf{V}}_r = \vec{\mathbf{V}}_a - \vec{\mathbf{V}}_e$



The track is horizontal :

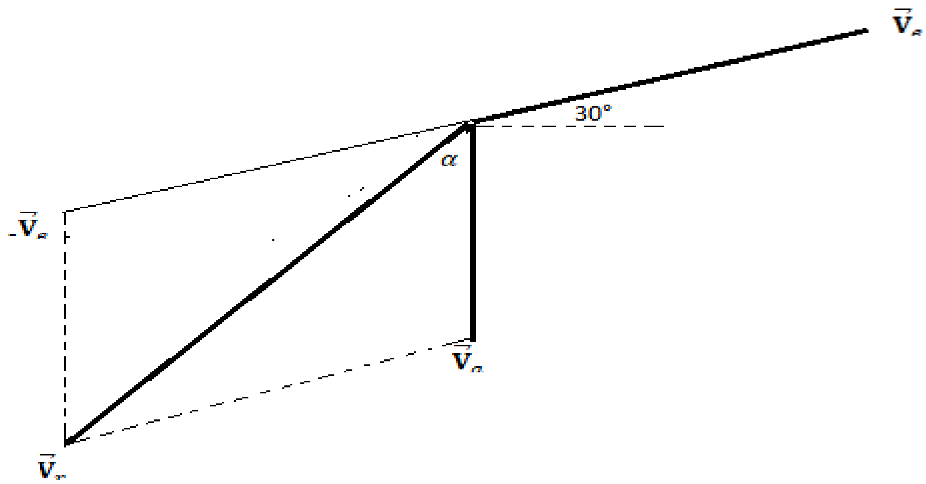
$\vec{\mathbf{V}}_a = -30\vec{\jmath}$ and $\vec{\mathbf{V}}_e = 40\vec{\imath} \rightarrow \quad \vec{\mathbf{V}}_r = -40\vec{\imath} - 30\vec{\jmath}$

$\left|\vec{\mathbf{V}}_r\right| = \sqrt{40^2 + 30^2} = 50\ m/s$

Flakes fall at an angle : $\alpha = arctan\frac{V_e}{V_a} = arctan\frac{4}{3} \approx\ 53.13^0$

- With what speed do these snowflakes hit the windshield of a car traveling with a constant speed of 40 m/s on a track with an upward inclination of an angle of 30° :

- $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r \ \rightarrow\ \vec{\mathbf{V}}_r\ = \vec{\mathbf{V}}_a - \vec{\mathbf{V}}_e$



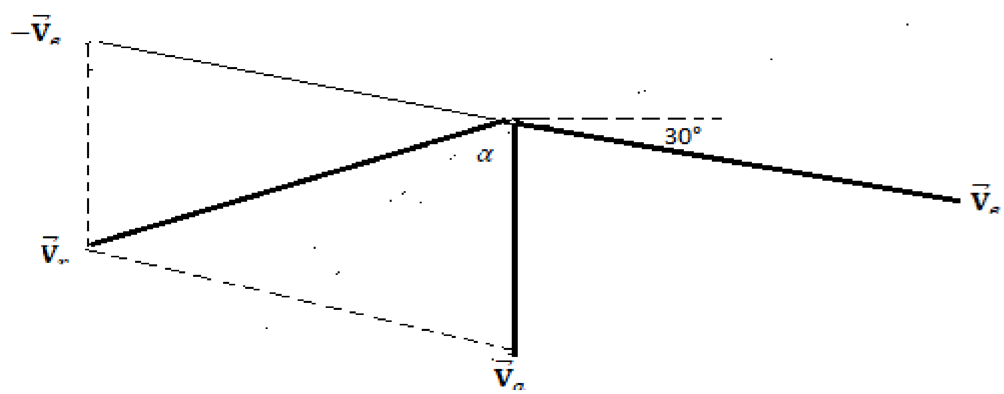
$\vec{\mathbf{V}}_a = -30\vec{\jmath}$ and $\vec{\mathbf{V}}_e = 40cos30\vec{\imath} + 40sin30\vec{\jmath} \ \rightarrow\ \vec{\mathbf{V}}_r\ = -20\sqrt{3}\vec{\imath} - 50\vec{\jmath}$

$\left|\vec{\mathbf{V}}_r\right| = 10\sqrt{37}\ m/s$

Flakes fall at an angle : $\alpha = arctan\frac{2\sqrt{3}}{5} \approx\ 34.72^0$

- With what speed do these snowflakes hit the windshield of a car traveling with a constant speed of 40 m/s on a track with a downward inclination of an angle of 30° :

- $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r \ \rightarrow\ \vec{\mathbf{V}}_r\ = \vec{\mathbf{V}}_a - \vec{\mathbf{V}}_e$



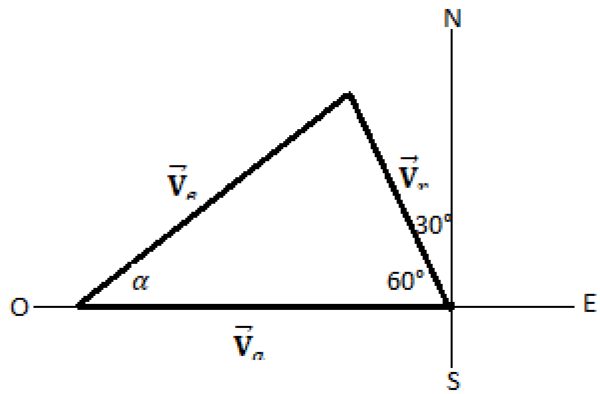
$\vec{\mathbf{V}}_a = -30\vec{\jmath}$ and $\vec{\mathbf{V}}_e = -40cos30\vec{\imath} - 40sin30\vec{\jmath} \ \rightarrow\ \vec{\mathbf{V}}_r\ = -20\sqrt{3}\vec{\imath} - 10\vec{\jmath}$

$\left|\vec{\mathbf{V}}_r\right| = 10\sqrt{13}\ m/s$

Flakes fall at an angle : $\alpha = arctan2\sqrt{3} \approx\ 73.90^0$

9. The movement of a ship is in the West direction relative to the land at a speed of 10 km/ hr. Its movement takes place in the direction of North 30° West at a speed of 8 km/ hr relative to the water.
   - Calculate the speed and direction of the water current relative to the land.

- $\vec{\mathbf{V}}_a$ is the absolute speed of the boat with respect to the land, $\vec{\mathbf{V}}_e$ is the speed of the water current with respect to the land, and $\vec{\mathbf{V}}_r$ is the speed of the boat with respect to the water :
  $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_e + \vec{\mathbf{V}}_r \rightarrow \vec{\mathbf{V}}_e = \vec{\mathbf{V}}_a - \vec{\mathbf{V}}_r$



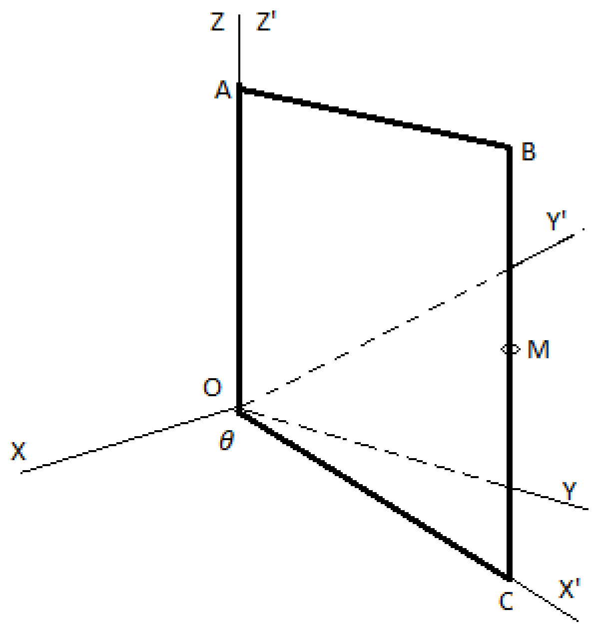
$\rightarrow \quad |\vec{\mathbf{V}}_e| = \sqrt{|\vec{\mathbf{V}}_a|^2 + |\vec{\mathbf{V}}_r|^2 - 2|\vec{\mathbf{V}}_a||\vec{\mathbf{V}}_r|cos60} = \sqrt{84}$km/hr

The direction of the drive velocity vector $\vec{\mathbf{V}}_e$ is defined by the angle α such that :

$\frac{|\vec{\mathbf{V}}_e|}{sin60} = \frac{|\vec{\mathbf{V}}_r|}{sin\alpha} \quad \rightarrow \quad sin\alpha = \frac{|\vec{\mathbf{V}}_r|}{|\vec{\mathbf{V}}_e|}sin60 = \frac{2}{\sqrt{7}} \quad \rightarrow \quad \alpha = \arcsin\left(\frac{2}{\sqrt{7}}\right)$

$\alpha \approx 49.11^0$

10. A square OABC of side L rotates around its side OA at constant angular velocity ω . A point material M moves along BC from B with a constant acceleration **a** and an initial speed V_o at point B.



- Calculate the absolute velocity and the absolute acceleration of the material point M.
- Determine the relative velocity, the training velocity of the material point M.
- Calculate its absolute velocity.
- Determine the relative acceleration, training acceleration and Coriolis acceleration.

- Deduce its absolute acceleration.

- The position vector relative to the absolute coordinate system :

  $\overrightarrow{\mathbf{OM}} = \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CM}} = L\vec{\imath}' + (L - x)\vec{\mathbf{k}}$ with $x = \frac{1}{2}At^2 + V_0 t$

  Using : $\frac{d\vec{\imath}'}{dt} = \omega\vec{\jmath}'$ and $\frac{d\vec{\jmath}'}{dt} = -\omega\vec{\imath}'$

  Absolute velocity $\vec{\mathbf{V}}_a = \frac{d\overrightarrow{\mathbf{OM}}}{dt} = L\frac{d\vec{\imath}'}{dt} - (At + V_0)\vec{\mathbf{k}} = L\omega\vec{\jmath}' - (At + V_0)\vec{\mathbf{k}}$

  Absolute acceleration $\vec{\mathbf{a}}_a = \frac{d\overrightarrow{\mathbf{V}_a}}{dt} = -L\omega^2\vec{\imath}' - A\vec{\mathbf{k}}$

- Relative velocity $\vec{\mathbf{V}}_r = \frac{d\overrightarrow{\mathbf{O'M}}}{dt} = \frac{d\overrightarrow{\mathbf{OM}}}{dt} = -(At + V_0)\vec{\mathbf{k}}$

  The training velocity $\vec{\mathbf{V}}_e = \frac{d\overrightarrow{\mathbf{O'O}}}{dt} + \vec{\omega}\wedge\overrightarrow{\mathbf{O'M}} = L\omega\vec{\jmath}'$

- Absolute velocity $\vec{\mathbf{V}}_a = \vec{\mathbf{V}}_r + \vec{\mathbf{V}}_e = L\omega\vec{\jmath}' - (At + V_0)\vec{\mathbf{k}}$**.** We find the same result.

- Relative acceleration $\vec{\mathbf{a}}_r = \frac{d\vec{\mathbf{V}}_r}{dt} = -A\vec{\mathbf{k}}$

  The training acceleration $\vec{\mathbf{a}}_e = \frac{d^2\overrightarrow{\mathbf{O'O}}}{dt^2} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{\mathbf{O'M}}\right) + \frac{\mathrm{d}\vec{\omega}}{\mathrm{dt}}\wedge\overrightarrow{\mathbf{O'M}} = \omega\vec{\mathbf{k}}\wedge L\omega\vec{\jmath}' = -L\omega^2\vec{\imath}'$

  Coriolis acceleration $\vec{\mathbf{a}}_c = 2\vec{\omega}\wedge\vec{\mathbf{V}}_r = 0$

  Absolute acceleration $\vec{\mathbf{a}}_a = \vec{\mathbf{a}}_r + \vec{\mathbf{a}}_e + \vec{\mathbf{a}}_c = -L\omega^2\vec{\imath}' - A\vec{\mathbf{k}}$**.** We find the same result.

Chapter 4

Dynamics

Dynamics is the analysis of the relationship between forces applied to a body and changes in the motion of that body. It explains the relationship that exists between the forces and the other kinematic quantities.

4.1 Galilean frame:

A Galilean frame is a frame consisting of a self-free system at rest or in uniform rectilinear motion.
Any reference frame in uniform rectilinear translation with respect to an other Galilean reference frame is itself Galilean.

4.2 Force:

A force is defined by a line of action, a direction, a point of application and an intensity.
There are two types of forces :

- Remote interaction forces such as :
  Gravitationnal force : $F = G\frac{mM}{r^2}$
  Electric force : $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$, $F = k\frac{qQ}{r^2}$
  Magnetic Force : $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\wedge\vec{\boldsymbol{B}}$, $F = \mu_0\frac{I_1 I_2 l}{2\pi r}$
- Contact forces such as :
  Friction force : $F = \mu N$
  Elastic forces : $F = K(L - L_0)$

4.3 Momentum:

The momentum of a particle is a vector quantity defined by the product of its mass and its instantaneous velocity vector : $\vec{\mathbf{p}} = \mathbf{m}\vec{\mathbf{v}}$
The momentum of an isolated system is conserved.
A free particle always moves with a constant momentum.

4.4 Fundamental law of dynamics:

The fundamental law of dynamics is given by Newton's laws :

- Principle of inertia : In a Galilean reference frame, the center of inertia of any mechanically isolated material system is either at rest or in uniform rectilinear motion.
- Fundamental principle of dynamics : In a Galilean reference frame, the vector sum of the forces applied to a point M of mass **m** and its acceleration $\vec{\mathbf{a}}$ are linked by : $\mathbf{m}\vec{\mathbf{a}} = \sum_i \vec{F_i}$
- Principle of action and reaction : The force exerted by a first body on a second body is equal and opposite to the force exerted by the second on the first : $\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}$

4.5 Generalized Fundamental principle of dynamics:

The variation of the momentum of a body with respect to time is equal to the resultant of the external forces applied :

$$\frac{d\vec{\mathbf{p}}}{dt} = \sum_i \vec{F_i}$$

4.6 The moment of a force:

The moment of a force applied to a material point located at point M, with respect to the fixed point O is defined by :

$$\vec{\mathbf{M}}_o = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{F}}$$

4.7 The angular momentum:

The angular momentum of a material point of mass **m** and velocity $\vec{\boldsymbol{v}}$, with respect to the point O is defined by :

$$\vec{\mathbf{L}}_o = \overrightarrow{\mathrm{OM}} \wedge m\vec{\mathbf{v}} = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{p}}$$

4.8 The angular momentum theorem:

The derivative of the angular momentum of a material point M with respect to a fixed point O in a Galilean reference frame is equal to the sum of the moments with respect to the same point O of the external forces applied to the point M :

$$\frac{d\vec{\mathbf{L}}_o}{dt} = \sum_i \vec{\mathbf{M}}_{i/o}$$

Exercises :

1. A body (M) of mass m moves along the position vector :

   $\vec{\boldsymbol{r}} = (t^2 + 3t)\vec{\imath} - t^3\vec{\jmath} + (3t - 1)\vec{k}$

   - Find the force $\vec{\boldsymbol{F}}$ acting on the body.
   - The moment $\vec{\boldsymbol{\tau}}$ of $\vec{\boldsymbol{F}}$ with respect to the origin O.
   - The momentum of $\vec{\boldsymbol{p}}$ the body and its angular momentum $\vec{\boldsymbol{L}}$ with respect to the origin O.
   - Check that : $\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt}$ and $\vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt}$

   - According to the fundamental principle of dynamics:

     $\vec{\boldsymbol{F}} = m\vec{a} = m\frac{d^2\vec{\boldsymbol{r}}}{dt^2} = m(2\vec{\imath} - 6t\vec{\jmath})$

   - The moment $\vec{\boldsymbol{\tau}}$ of $\vec{\boldsymbol{F}}$ with respect to the origin O is equal :

     $$\vec{\boldsymbol{\tau}} = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{F}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ (t^2 + 3t) & -t^3 & (3t - 1) \\ 2m & -6mt & 0 \end{vmatrix}$$

     $\vec{\boldsymbol{\tau}} = 6mt(3t - 1)\vec{\imath} + 2m(3t - 1)\vec{\jmath} - 2mt^2(2t + 9)\vec{k}$

   - The momentum $\vec{\boldsymbol{p}}$ is equal :

     $$\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}} = m\frac{d\vec{\boldsymbol{r}}}{dt} = m[(2t + 3)\vec{\imath} - 3t^2\vec{\jmath} + 3\vec{k}]$$

   - The angular momentum $\vec{\boldsymbol{L}}$with respect to the origin O is equal :

     $$\vec{\mathbf{L}}_o = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{p}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ (t^2 + 3t) & -t^3 & (3t - 1) \\ m(2t + 3) & -3mt^2 & 3m \end{vmatrix}$$

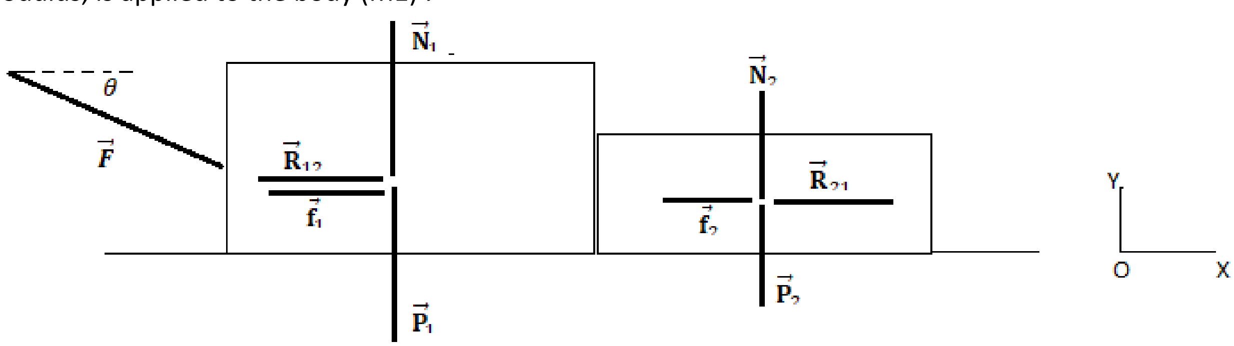
     $\vec{\mathbf{L}}_o = 3mt^2(2t - 3)\vec{\imath} + m(3t^2 - 2t - 3)\vec{\jmath} - 2m(2t + 9)\vec{k}$

   - We can verify that :

     $\frac{d\vec{\boldsymbol{p}}}{dt} = m[2\vec{\imath} - 6t\vec{\jmath}]$ is equal to the force $\vec{\boldsymbol{F}}$.

     $\frac{d\vec{\boldsymbol{L}}}{dt} = 6mt(3t - 1)\vec{\imath} + 2m(3t - 1)\vec{\jmath} - 2mt^2(2t + 9)\vec{k}$ is equal to moment $\vec{\boldsymbol{\tau}}$.

2. Two bodies (M1) of mass $\mathbf{m_1}$ and (M2) of mass $\mathbf{m_2}$ are placed in contact on a horizontal table. The static and dynamic coefficients of friction between the bodies (M1), (M2) and the contact surface are respectively μ_s and μ_d. A force $\vec{\boldsymbol{F}}$ forming an angle θ with the horizontal and of constant modulus, is applied to the body (M1) :



- Determine the maximum modulus of force $\vec{F}$ required to move the two bodies.
- Write the fundamental dynamic relation for each mass in the case of motion.
- Deduce the acceleration $\vec{a}$ of the system.
- Determine the contact force between the two bodies.

- The mass (M1) being in static state is subjected to the forces: $\vec{F}$, its weight $\vec{\mathbf{P}}_1$, the reaction of the table $\vec{\mathbf{N}}_1$, the reaction of the mass (M2) $\vec{\mathbf{R}}_{12}$ and the force of static friction $\vec{\mathbf{f}}_{s1}$:

  $\vec{F} + \vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{R}}_{12} + \vec{\mathbf{f}}_{s1} = 0$

  The mass (M2) being also in static state is subjected to the forces : its weight $\vec{\mathbf{P}}_2$, the reaction of the table $\vec{\mathbf{N}}_2$, the reaction of the mass (M1) $\vec{\mathbf{R}}_{21}$and the force of static friction $\vec{\mathbf{f}}_{s2}$:

  $\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{R}}_{21} + \vec{\mathbf{f}}_{s2} = 0$

  By projecting these two equations onto the horizontal OX and vertical OY axes :

  OX : $Fcos\theta - R_{12} - f_{s1} = 0$

  $R_{21} - f_{s2} = 0$

  OY : $-Fsin\theta - P_1 + N_1 = 0$

  $-P_2 + N_2 = 0$

  Principle of action and reaction : $R_{12} = R_{21}$

  $f_{s1} = \mu_s N_1 = \mu_s(Fsin\theta + P_1) = \mu_s(Fsin\theta + m_1 g)$

  and $f_{s2} = \mu_s N_2 = \mu_s P_2 = \mu_s m_2 g$

  $Fcos\theta - \mu_s(Fsin\theta + m_1 g) - \mu_s m_2 g = 0$

  So : $F = \frac{\mu_s}{cos\theta - \mu_s sin\theta}(m_1 + m_2)g$

- the fundamental relation of the dynamics for each mass in the case of motion :

  The mass (M1) : $\vec{F} + \vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{R}}_{12} + \vec{\mathbf{f}}_{d1} = m_1 \vec{\mathbf{a}}_1$

  The mass (M2) : $\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{R}}_{21} + \vec{\mathbf{f}}_{d2} = m_2 \vec{\mathbf{a}}_2$

  By projecting these two equations onto the horizontal OX and vertical OY axes :

  OX : $Fcos\theta - R_{12} - f_{d1} = m_1 a_1$

  $R_{21} - f_{d2} = m_2 a_2$

  OY : $-Fsin\theta - P_1 + N_1 = 0$

  $-P_2 + N_2 = 0$

  $f_{d1} = \mu_d N_1 = \mu_d(Fsin\theta + P_1) = \mu_d(Fsin\theta + m_1 g)$

  and $f_{d2} = \mu_d N_2 = \mu_d P_2 = \mu_d m_2 g$

  Principle of action and reaction : $R_{12} = R_{21}$

  The mass system (M1) and (M2) moves with the same acceleration : $a_1 = a_2 = a$

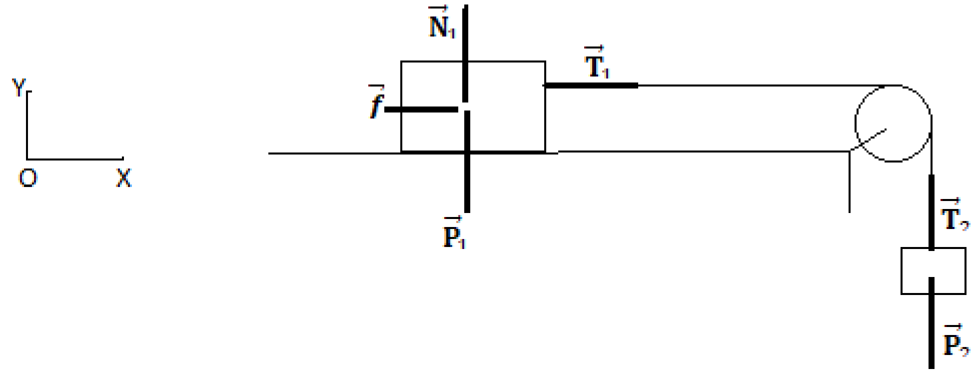
  $Fcos\theta - \mu_d(Fsin\theta + m_1 g) - \mu_d m_2 g = (m_1 + m_2)a$

  So the acceleration of the two masses is : $a = \frac{F(cos\theta - \mu_d sin\theta) - \mu_d(m_1 + m_2)g}{(m_1 + m_2)}$

- The force of contact between the two bodies $R_{12} = R_{21} = R$

  $R = m_2 a + f_{d2} = m_2(a + \mu_d g) = \frac{m_2}{(m_1 + m_2)}(cos\theta - \mu_d sin\theta)F$

3. Two bodies (M1) of mass $\mathbf{m_1}$ and (M2) of mass $\mathbf{m_2}$ are connected by an inextensible wire passing through a pulley of negligible mass (see figure). The static and dynamic coefficients of friction between the body (M1) and the contact surface are μ_s and μ_d respectively.



- Study the condition of static equilibrium of the two bodies (M1) and (M2).
- Study the motion of the system.
- Calculate the tension of the thread connecting the two bodies.

- The mass (M1) being in static state is subjected to the forces : its weight $\vec{\mathbf{P}}_1$, the reaction of the contact surface $\vec{\mathbf{N}}_1$, the tension of the wire $\vec{\mathbf{T}}_1$ and the force of static friction $\vec{\mathbf{f}}_s$:

  $\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_s = 0$

  The mass (M2) being in static state is subjected its weight $\vec{\mathbf{P}}_2$ and the tension of the wire $\vec{\mathbf{T}}_2$:

  $\vec{\mathbf{P}}_2 + \vec{\mathbf{T}}_2 = 0$

  By projecting these two equations onto the horizontal OX and vertical OY axes :

  OX : $T_1 - f_s = 0$

  OY : $-P_1 + N_1 = 0$

  $-P_2 + T_2 = 0$

  The wire is inextensible and the pulley is of negligible mass $\rightarrow$ $T_1 = T_2 = T$

  $f_s = \mu_s N_1 = \mu_s m_1 g$

  $\rightarrow$ $m_2 g - f_s = 0$ $\rightarrow m_2 - \mu_s m_1 = 0$

  Therefore, the static equilibrium condition of the two bodies (M1) and (M2) is :

  $m_2 = \mu_s m_1$

- the fundamental principle of dynamics for each mass in the case of motion :

  The mass (M1) : $\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_d = m_1 \vec{\mathbf{a}}_1$

  The mass (M2) : $\vec{\mathbf{P}}_2 + \vec{\mathbf{T}}_2 = m_2 \vec{\mathbf{a}}_2$

  By projecting these two equations onto the horizontal OX and vertical OY axes :

  OX : $T_1 - f_d = m_1 a_1$

  OY : $-m_1 g + N_1 = 0$

  $-m_2 g + T_2 = -m_2 a_2$

  The wire is inextensible and the pulley is of negligible mass $\rightarrow$ $T_1 = T_2 = T$ and $a_1 = a_2 = a$

  $f_d = \mu_d N_1 = \mu_d m_1 g$

  $(m_2 - \mu_d m_1) g = (m_1 + m_2) a$

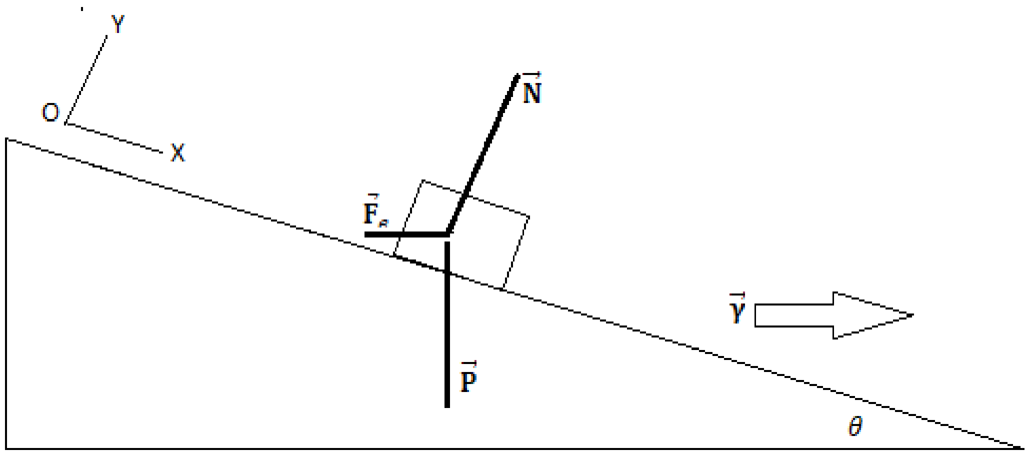
  Therefore, the system acceleration is :

$$a = \frac{(m_2 - \mu_d m_1)}{(m_1 + m_2)} g$$

- The thread tension T is :

$$T = m_2(g - a) = \frac{m_1 m_2 (1 + \mu_d)}{(m_1 + m_2)} g$$

4. A material point (M) of mass **m** moves without friction on the inclined surface of a block (B). This last moves on the horizontal with a constant acceleration $\vec{\boldsymbol{\gamma}}$. Knowing that the initial velocity of the material point (M) is zero with respect to the block (B) :



- Study the motion of (M) in the moving frame **R'**.
- Determine the reaction of the block (B) on the mass m.
- Deduce the acceleration $\vec{\boldsymbol{\gamma}}$ for the material point (M) to rise from the inclined plane.

- By applying the fundamental principle of dynamics in the moving frame **R'** :

$\vec{\mathbf{P}} + \vec{\mathbf{N}} + \vec{\mathbf{F}}_e + \vec{\mathbf{F}}_c = m\vec{\mathbf{a}}_r$

where $\vec{\mathbf{F}}_e = -m\vec{\boldsymbol{\gamma}}$ is the driving inertial force and $\vec{\mathbf{F}}_c = 0$ the Coriolis inertial force.

By projecting on the OX and OY axes :

$mg\sin\theta - m\gamma\cos\theta = ma_r \quad \rightarrow \quad a_r = g\sin\theta - \gamma\cos\theta$

$-mg\cos\theta + N - m\gamma\sin\theta = 0$

- The reaction of the block (B) on the mass m : $N = m(g\cos\theta + \gamma\sin\theta)$
- The material point (M) rises from the inclined plane $\rightarrow \quad N = 0$

$g\cos\theta + \gamma\sin\theta = 0 \quad \rightarrow \quad \gamma = -g\cot an\theta$

The block (B) must move with an acceleration directed to the left.

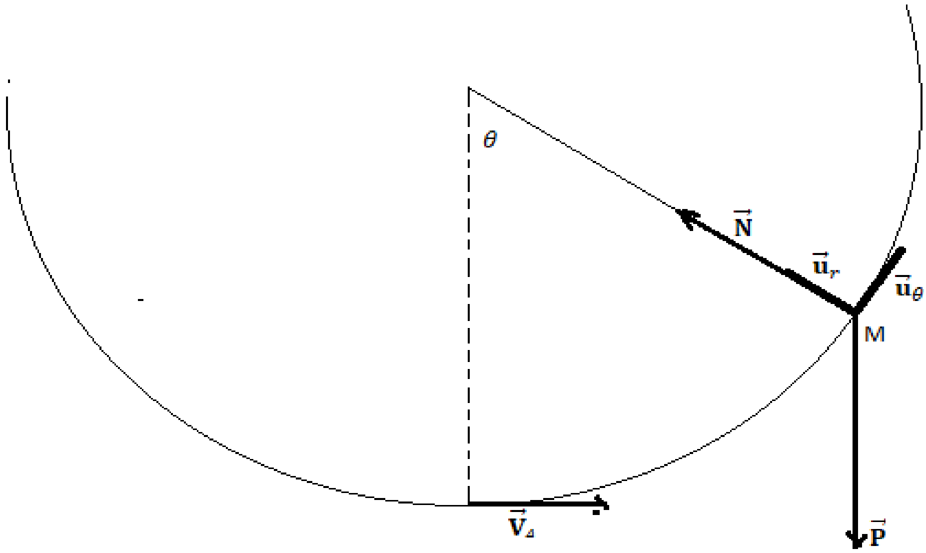
5. A material point (M) of mass m moves without friction inside a circular surface of radius R and center O. The mass m is launched from A with an initial velocity v_A.

Using the Fundamental Principle of Dynamics :

- Determine the expression for the speed of the mass m.
- Determine at what angle the velocity vanishes.
- Deduce the expression of the reaction $\vec{N}$ of the surface on the material point (M).

Applying the angular momentum theorem :

- Find a differential equation governing θ (t).



- By applying the fundamental principle of dynamics :

  $\vec{\mathbf{P}} + \vec{\mathbf{N}} = m\vec{\mathbf{a}}$

  Using polar coordinates :

  $m\left[(\ddot{r} - r\dot{\theta}^2)\vec{\mathbf{u}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{\mathbf{u}}_\theta\right] = (mgcos\theta - N)\vec{\mathbf{u}}_r - mgsin\theta\vec{\mathbf{u}}_\theta$

  Since $r = R \quad \rightarrow \quad \dot{r} = \ddot{r} = 0$

  $$\rightarrow \quad \begin{cases} mR\dot{\theta}^2 = -mgcos\theta + N \\ R\ddot{\theta} = -gsin\theta \end{cases}$$

  The velocity $v = R\frac{d\theta}{dt} = R\dot{\theta} \quad \rightarrow \quad \frac{dv}{dt} = R\ddot{\theta} \quad \rightarrow \quad vdv = -gRsin\theta d\theta$

  $$\int_{c_A}^{v} vdv = \int_0^{\theta} -gRsin\theta d\theta \quad \rightarrow \quad \frac{1}{2}(v^2 - {v_A}^2) = gR(cos\theta - 1)$$

  The expression for the velocity is then :

  $v = \sqrt{{v_A}^2 - 2gR(1 - cos\theta)}$

- The object stops and velocity vanishes : $v = 0 \quad \rightarrow \quad cos\theta = 1 - \frac{{v_A}^2}{2gR}$

  $\theta = arcos(1 - \frac{{v_A}^2}{2gR})$

- The reaction $\vec{\boldsymbol{N}}$ of the surface on the material point (M) :

  $N = mR\dot{\theta}^2 + mgcos\theta = m\frac{v^2}{R} + mgcos\theta$

  $$N = m\frac{{v_A}^2 - 2gR(1 - cos\theta)}{R} + mgcos\theta = m\frac{{v_A}^2 - gR(2 - 3cos\theta)}{R}$$

- The angular momentum theorem : $\vec{\mathbf{L}}_o = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{p}} = \overrightarrow{\mathrm{OM}} \wedge m\vec{\mathrm{v}}$

  $\overrightarrow{\mathrm{OM}} = -R\vec{\mathbf{u}}_r \quad et \quad \vec{\mathrm{v}} = v\vec{\mathbf{u}}_\theta$

  So $\vec{\mathbf{L}}_o = -mRv\vec{\mathbf{k}}$

  The moment of forces : $\vec{\mathbf{M}}_o = \overrightarrow{\mathrm{OM}} \wedge \vec{\mathbf{F}}$

  $\vec{\mathbf{M}}_o = \overrightarrow{\mathrm{OM}} \wedge (\vec{\mathbf{P}} + \vec{\mathbf{N}}) = -R\vec{\mathbf{u}}_r \wedge (mgcos\theta - N)\vec{\mathbf{u}}_r - mgsin\theta\vec{\mathbf{u}}_\theta = mgRsin\theta\vec{\mathbf{k}}$

$$\frac{d\vec{\mathbf{L}}_o}{dt} = \sum_i \vec{\mathbf{M}}_{i/o} \quad \rightarrow -mR\frac{dv}{dt}\vec{\mathbf{k}} = mgRsin\theta\vec{\mathbf{k}} \quad \rightarrow \quad \frac{dv}{dt} = -gsin\theta$$

We find the same result.

6. A particle of charge q and mass m, moving with a velocity $\vec{\boldsymbol{v}}$ in an electromagnetic field : an electric field $\vec{\boldsymbol{E}} = E\vec{\boldsymbol{\imath}}$ and a magnetic field $\vec{\boldsymbol{B}} = B\vec{\boldsymbol{\jmath}}$, undergoes a force of the form : $\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}}\wedge\vec{\boldsymbol{B}})$.
   We assume $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ constants in modulus and direction. Initially, the particle is at the origin with zero initial velocity.
   - Study the motion of the particle and find the differential equation of motion.
   - By applying the fundamental principle of dynamics for the particle:

$$m\vec{\boldsymbol{a}} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}}\wedge\vec{\boldsymbol{B}}) \ \rightarrow \ m(\ddot{x}\vec{\boldsymbol{\imath}} + \ddot{y}\vec{\boldsymbol{\jmath}} + \ddot{z}\vec{\boldsymbol{k}}) = q[E\vec{\boldsymbol{\imath}} + (\dot{x}\vec{\boldsymbol{\imath}} + \dot{y}\vec{\boldsymbol{\jmath}} + \dot{z}\vec{\boldsymbol{k}})\wedge B\vec{\boldsymbol{\jmath}}]$$

$$m(\ddot{x}\vec{\boldsymbol{\imath}} + \ddot{y}\vec{\boldsymbol{\jmath}} + \ddot{z}\vec{\boldsymbol{k}}) = q[E\vec{\boldsymbol{\imath}} + B(\dot{x}\vec{\boldsymbol{k}} - \dot{y}\vec{\boldsymbol{\imath}})] = q\{(E - B\dot{y})\vec{\boldsymbol{\imath}} + B\dot{x}\vec{\boldsymbol{k}}\}$$

$$\begin{cases} m\ddot{x} = q(E - B\dot{z}) \\ m\ddot{y} = 0 \\ m\ddot{z} = qB\dot{x} \end{cases} \rightarrow \begin{cases} \ddot{x} = \frac{q}{m}(E - B\dot{z}) \\ \ddot{y} = 0 \\ \ddot{z} = \frac{q}{m}B\dot{x} \end{cases}$$

Because $x(0) = y(0) = z(0) = 0$ et $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$

$y(t) = 0$

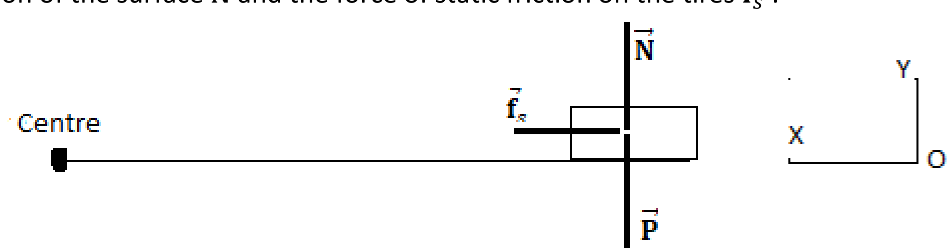
$$\dot{z} = \frac{q}{m}Bx \quad \rightarrow \quad \ddot{x} = \frac{q}{m}\left(E - \frac{q}{m}B^2x\right) \quad \rightarrow \quad \ddot{x} + (\frac{qB}{m})^2x = \frac{q}{m}E$$

It can be shown that the solutions of the differential equations x(t) and z(t) are :

$$x(t) = -\frac{mE}{qB^2}\cos\left(\frac{qB}{m}t\right) + \frac{mE}{qB^2}$$

$$z(t) = -\frac{mE}{qB^2}\sin\left(\frac{qB}{m}t\right) + \frac{E}{B}t$$

7. An automobile with a mass of 1000 kg enters a circular curve of radius R = 100 m.
   - If the road is not inclined, what must be the coefficient of static friction between the tires and the road to prevent the car from sliding with a speed v = 25 m/s.
   - What is the inclination of the roadway to allow a speed of 25 m/s in all weathers (No friction).
   - If the pavement is raised by 30^0 above the horizontal and the coefficient of static friction is $\mu_s = 0.1$, find the value of the maximum speed at which the automobile can travel without risk.
   - The pavement being horizontal, the automobile is subjected to the forces: its weight $\vec{\mathbf{P}}$, the reaction of the surface $\vec{\mathbf{N}}$ and the force of static friction on the tires $\vec{\mathbf{f}}_s$:



By applying the fundamental principle of dynamics for the automobile :

$$\vec{\mathbf{P}} + \vec{\mathbf{N}} + \vec{\mathbf{f}}_s = m\vec{\mathbf{a}}$$

The motion being uniform and circular, the acceleration $\vec{\mathbf{a}}$ is oriented towards the center of the curve with a magnitude $a = \frac{v^2}{R}$

By projecting the equations on the radial OX and vertical OY axes :

OX : $f_s = ma$

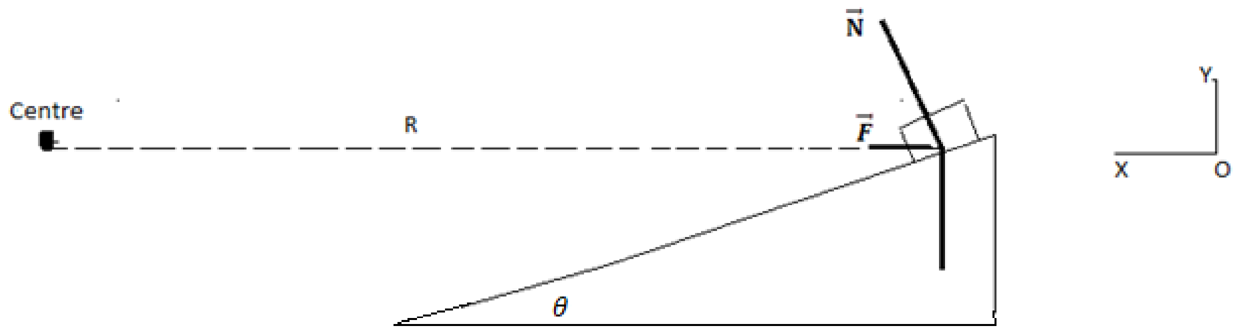
OY : $-P + N = 0$

$f_s = \mu_s N = \mu_s mg$

Therefore $\mu_s mg = m\frac{v^2}{R} \;\; \rightarrow \;\; \mu_s = \frac{v^2}{Rg}$

N.A : $\mu_s = 0.625$

- In the absence of friction, the pavement is inclined at an angle θ :



The automobile is subject to the forces of its weight $\vec{\mathbf{P}}$ and the reaction of the road surface $\vec{\mathbf{N}}$:

By applying the fundamental principle of dynamics :

$\vec{\mathbf{P}} + \vec{\mathbf{N}} = m\vec{\mathbf{a}}$

The movement being uniform and circular, the acceleration $\vec{\mathbf{a}}$ is oriented towards the center of the curve with a module $a = \frac{v^2}{R}$

By projecting the equations on the horizontal and radial OX and vertical OY axes :

OX : $Nsin\theta = ma$

OY : $-P + Ncos\theta = 0$

$\rightarrow \; N = \frac{mg}{cos\theta} \;\; \rightarrow \;\; mgtan\theta = ma = m\frac{v^2}{R}$

$tan\theta = \frac{v^2}{Rg} \;\; \rightarrow \;\; \theta = \arctan\left(\frac{v^2}{Rg}\right)$

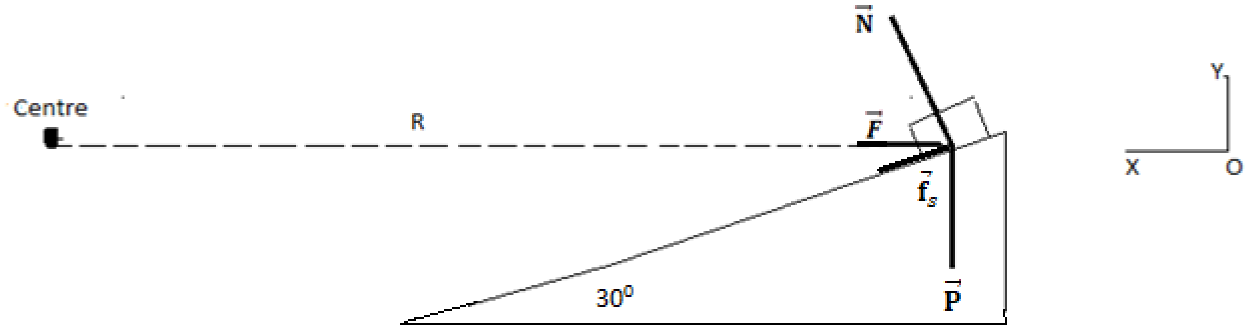
N.A : The inclination $\theta \approx 33^0$

- The pavement being raised and the static friction on the tires is present : The automobile is subjected to the forces : its weight $\vec{\mathbf{P}}$, the reaction of the surface $\vec{\mathbf{N}}$ and the force of static friction on the tires $\vec{\mathbf{f}}_s$:

By applying the fundamental principle of dynamics for the automobile :

$\vec{\mathbf{P}} + \vec{\mathbf{N}} + \vec{\mathbf{f}}_s = m\vec{\mathbf{a}}$

The movement being uniform and circular, the acceleration $\vec{\mathbf{a}}$ is oriented towards the center of the curve with a module $a = \frac{v^2}{R}$



By projecting the equations on the radial OX and vertical OY axes :

OX : $Nsin\theta + f_s cos\theta = ma$

OY : $-P + Ncos\theta - f_s sin\theta = 0$

$f_s = \mu_s N$

$\rightarrow \quad N = \frac{mg}{cos\theta - \mu_s sin\theta}$

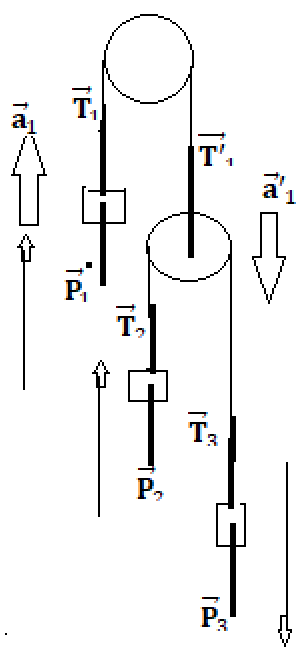
$\rightarrow \quad v^2 = \frac{sin\theta + \mu_s cos\theta}{cos\theta - \mu_s sin\theta} Rg$

So the maximum speed at which the automobile can drive without risk is :

$$v = \sqrt{\frac{sin\theta + \mu_s cos\theta}{cos\theta - \mu_s sin\theta} Rg}$$

N.A : $v \approx 26.54\, m/s$

8. Find the accelerations of the bodies in the figure below neglecting the frictional forces, the masses of the pulleys and those of the wires which we consider as inextensible.



- By applying the fundamental principle of dynamics for the three masses separately :

  Mass M1 : $\vec{P}_1 + \vec{T}_1 = m_1 \vec{a}_1$

By projecting on the ascending axis : $-m_1 g + T_1 = m_1 a_1$

Mass M2 : $\vec{\mathbf{P}}_2 + \vec{\mathbf{T}}_2 = m_2 \vec{\mathbf{a}}_2$

Acceleration $\vec{\mathbf{a}}_2 = \vec{\mathbf{a}}_{2r} + \vec{\mathbf{a}}_e$ where $\vec{\mathbf{a}}_e = \vec{\mathbf{a}}'_1 = -\vec{\mathbf{a}}_1$ is the training acceleration

By projecting on the ascending axis : $-m_2 g + T_2 = m_2(a_{2r} - a_1)$

Mass M3 : $\vec{\mathbf{P}}_3 + \vec{\mathbf{T}}_3 = m_3 \vec{\mathbf{a}}_3$

Acceleration $\vec{\mathbf{a}}_3 = \vec{\mathbf{a}}_{3r} + \vec{\mathbf{a}}_e$ where $\vec{\mathbf{a}}_e = \vec{\mathbf{a}}'_1 = -\vec{\mathbf{a}}_1$ is the training acceleration

By projecting on the descending axis : $m_3 g - T_3 = m_3(a_{3r} + a_1)$

Since the pulleys are of negligible mass and the wires are inextensible:

$\rightarrow\ a_{2r} = a_{3r} = a_r,\ T_1 = T'_1 = T_2 + T_3$ and $T_2 = T_3 = T$

The three projections become:

$$\begin{cases} -m_1 g + 2T = m_1 a_1 \\ -m_2 g + T = m_2(a_r - a_1) \\ m_3 g - T = m_3(a_r + a_1) \end{cases}$$

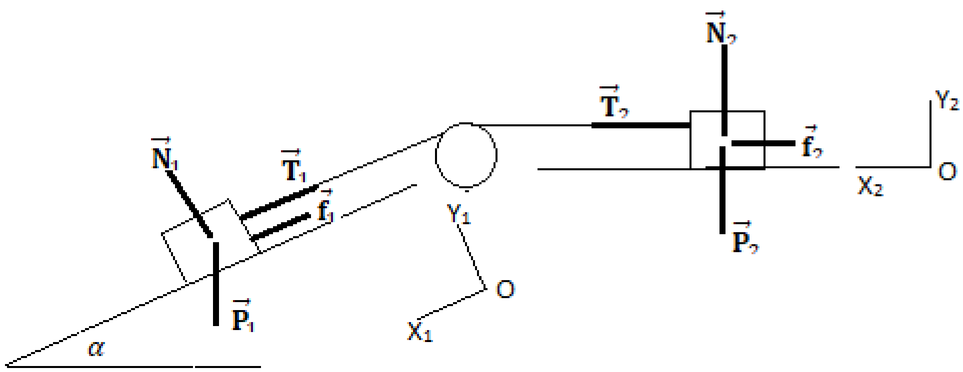
It is a system of three equations with three unknowns a_1, a_r and T :

$$\begin{cases} a_1 = \dfrac{4m_2 m_3 - m_1 m_2 - m_1 m_3}{m_1 m_2 + m_1 m_3 + 4m_2 m_3} \\ a_r = \dfrac{2(m_1 m_3 - m_1 m_2)}{m_1 m_2 + m_1 m_3 + 4m_2 m_3} \\ T = \dfrac{4m_2 m_3}{m_1 m_2 + m_1 m_3 + 4m_2 m_3} m_1 g \end{cases}$$

We notice that if $m_2 = m_3$; the relative acceleration a_r vanishes.

We also notice that if $m_1 = \frac{m_2 m_3}{m_2 + m_3}$; absolute acceleration a_1 vanishes.

9. Study the static case and the dynamic case of the two bodies in the figure below, neglecting the masses of the pulleys and those of the wires which we consider as inextensible.
The static and dynamic coefficients of friction between the bodies (M1) and (M2) and the surface of contact are μ_s and μ_d respectively.



- Static case :

The mass (M1) being in static state is subjected to the forces: its weight $\vec{\mathbf{P}}_1$, the reaction of the surface $\vec{\mathbf{N}}_1$, the tension $\vec{\mathbf{T}}_1$ and the force of static friction $\vec{\mathbf{f}}_{1s}$:

$\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_{1s} = 0$

By projecting on OX_1 and OY_1 axes :

OX$_1$: $m_1 g sin\alpha - T_1 - f_{1s} = 0$

OY$_1$: $-m_1 g cos\alpha + N_1 = 0 \ \rightarrow \ N_1 = m_1 g cos\alpha$

The mass (M2) being in static state is subjected to the forces : its weight $\vec{\mathbf{P}}_2$, the reaction of the surface $\vec{\mathbf{N}}_2$, the tension $\vec{\mathbf{T}}_2$ and the force of static friction $\vec{\mathbf{f}}_{2s}$:

$\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{T}}_2 + \vec{\mathbf{f}}_{2s} = 0$

By projecting on OX$_2$ and OY$_2$ axes :

OX$_2$: $T_2 - f_{2s} = 0$

OY$_2$: $-m_2 g + N_2 = 0 \ \rightarrow \ N_2 = m_2 g$

knowing that $f_{1s} = \mu_s N_1 = \mu_s m_1 g cos\alpha$ and $f_{2s} = \mu_s N_2 = \mu_s m_2 g$

and $T_1 = T_2 = T$ because the threads are massless and inextensible.

The static equilibrium condition is : $m_1(sin\alpha - \mu_s cos\alpha) = \mu_s m_2$

- Dynamic case :

  By applying the fundamental principle of dynamics for the two masses separately :

  Mass M1 : $\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_{1d} = m_1 \vec{\mathbf{a}}_1$

  By projecting on OX$_1$ and OY$_1$ axes :

  OX1 : $m_1 g sin\alpha - T_1 - f_{1d} = m_1 a_1$

  OY1 : $-m_1 g cos\alpha + N_1 = 0 \ \rightarrow \ N_1 = m_1 g cos\alpha$

  Mass M2 : $\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{T}}_2 + \vec{\mathbf{f}}_{2d} = m_2 \vec{\mathbf{a}}_2$

  By projecting on OX$_2$ and OY$_2$ axes :

  OX2 : $T_2 - f_{2d} = m_2 a_2$

  OY2 : $-m_2 g + N_2 = 0 \ \rightarrow \ N_2 = m_2 g$

  $f_{1d} = \mu_d N_1 = \mu_d m_1 g cos\alpha$ and $f_{2d} = \mu_d N_2 = \mu_d m_2 g$

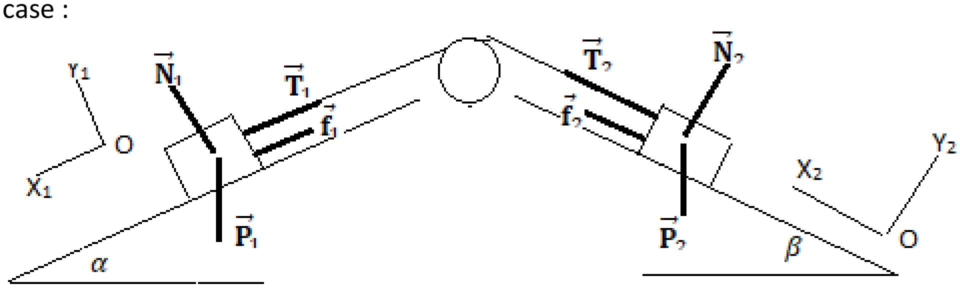
  Since the threads are massless and inextensible $\rightarrow$ $a_1 = a_2 = a$ and $T_1 = T_2 = T$

  The system acceleration is : $a = \frac{(sin\alpha - \mu_d cos\alpha)m_1 - \mu_d m_2}{m_1 + m_2} g$

  The thread tension is : $T = \frac{(sin\alpha - \mu_d cos\alpha + \mu_d) m_1 m_2}{m_1 + m_2} g$

10. Study the static case and the dynamic case of the two bodies of the figure below, neglecting the masses of the pulleys and those of the wires which we consider as inextensible.
    The static and dynamic coefficients of friction between the bodies (M1) and (M2) and the surface of contact are μ_s and μ_d respectively.

- Static case :



  The mass (M1) being in static state is subjected to the forces : its weight $\vec{\mathbf{P}}_1$, the reaction of the surface $\vec{\mathbf{N}}_1$, the tension $\vec{\mathbf{T}}_1$ and the force of static friction $\vec{\mathbf{f}}_{1s}$:

$\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_{1s} = 0$

By projecting on OX_1 and OY_1 axes :

OX1 : $m_1 g sin\alpha - T_1 - f_{1s} = 0$

OY1 : $-m_1 g cos\alpha + N_1 = 0 \rightarrow N_1 = m_1 g cos\alpha$

The mass (M2) being in static state is subjected to the forces : its weight $\vec{\mathbf{P}}_2$, the reaction of the surface $\vec{\mathbf{N}}_2$, the tension $\vec{\mathbf{T}}_2$ and the force of static friction $\vec{\mathbf{f}}_{2s}$:

$\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{T}}_2 + \vec{\mathbf{f}}_{2s} = 0$

By projecting on OX_2 and OY_2 axes :

OX2 : $-m_2 g sin\beta + T_2 + f_{2s} = 0$

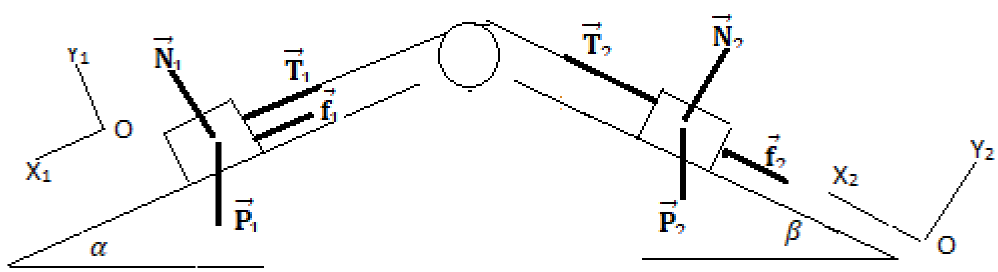
OY2 : $-m_2 g cos\beta + N_2 = 0 \rightarrow N_2 = m_2 g cos\beta$

knowing that $f_{1s} = \mu_s N_1 = \mu_s m_1 g cos\alpha$ and $f_{2s} = \mu_s N_2 = \mu_s m_2 g cos\beta$

and $T_1 = T_2 = T$ because the threads are massless and inextensible.

The static equilibrium condition is then : $m_1(sin\alpha - \mu_s cos\alpha) = m_2(sin\beta - \mu_s cos\beta)$

- Dynamic case: We are confronted with two cases :
  - $m_1(sin\alpha - \mu_s cos\alpha) > m_2(sin\beta - \mu_s cos\beta)$



$\rightarrow$ The mass M1 is descending and the mass M2 is ascending :

By applying the fundamental principle of dynamics for the two masses separately :

Mass M1: $\vec{\mathbf{P}}_1 + \vec{\mathbf{N}}_1 + \vec{\mathbf{T}}_1 + \vec{\mathbf{f}}_{1d} = m_1 \vec{\mathbf{a}}_1$

By projecting on OX_1 and OY_1 axes :

OX1 : $m_1 g sin\alpha - T_1 - f_{1d} = m_1 a_1$

OY1 : $-m_1 g cos\alpha + N_1 = 0 \rightarrow N_1 = m_1 g cos\alpha$

Mass M2: $\vec{\mathbf{P}}_2 + \vec{\mathbf{N}}_2 + \vec{\mathbf{T}}_2 + \vec{\mathbf{f}}_{2d} = m_2 \vec{\mathbf{a}}_2$

By projecting on OX_2 and OY_2 axes :

OX2 : $-m_2 g sin\beta + T_2 - f_{2d} = m_2 a_2$

OY2 : $-m_2 g cos\beta + N_2 = 0 \rightarrow N_2 = m_2 g cos\beta$

$f_{1d} = \mu_d N_1 = \mu_d m_1 g cos\alpha$ and $f_{2d} = \mu_d N_2 = \mu_d m_2 g cos\beta$

Since the pulley and wires are massless and inextensible $\rightarrow$ $a_1 = a_2 = a$ and $T_1 = T_2 = T$

The system acceleration is : $a = \frac{(sin\alpha - \mu_d cos\alpha)m_1 - (sin\beta + \mu_d cos\beta)m_2}{m_1 + m_2} g$

  - $m_1(sin\alpha - \mu_s cos\alpha) < m_2(sin\beta - \mu_s cos\beta)$

$\rightarrow$ The mass M1 is ascending and the mass M2 is descending :

By analogy to the first case :

The system acceleration is : $\tilde{a} = \frac{(sin\beta - \mu_d os\beta)m_1 - (sin\alpha + \mu_d cos\alpha)m_2}{m_1 + m_2} g$

We note that $a \neq -\tilde{a}$ and this is due to the presence of friction forces.

Chapter 5

Work and Energy

5.1 Work of a force:

The work of a force $\vec{F}$applied to a material point moving between two points A and B is :

$$W_{AB} = \int_{(A)}^{(B)} \vec{F} \cdot d\vec{l} = \int_{(A)}^{(B)} (F_x dx + F_y dy + F_z dz)$$

The work of a force $\vec{F}$ perpendicular to the displacement vector is zero.

The work of a force $\vec{F}$ constant in magnitude and direction is equal: $W_{AB} = F \cdot AB \cdot \cos(\vec{F}, \overrightarrow{AB})$

The unit of work is : joule (J) = Newton-meter (N ˙ m)

5.2 Power of a force:

The power of a force$\vec{F}$ at each instant is :

$$P = \vec{F} \cdot \vec{v}$$

The power of a force is also defined as the instantaneous variation of its work:

$$P = \frac{dW}{dt}$$

The unit of power is : Watt = joule/s

5.3 Conservative force:

A force is said to be conservative if its work between two points does not depend on the path followed.

Any conservative force derives from a potential function $E_p(x, y, z)$such that :

$$\vec{F} = -\overrightarrow{grad} E_p(x, yz) = \frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}$$

$$\rightarrow \begin{cases} F_x = -\frac{\partial E_p}{\partial x} \\ F_y = -\frac{\partial E_p}{\partial y} \\ F_z = -\frac{\partial E_p}{\partial z} \end{cases} \quad \rightarrow \quad W_{AB} = -\Delta E_p = E_p(A) - E_p(B)$$

If a force $\vec{F}$ is conservative, its rotational is zero : $\overrightarrow{rot}\,\vec{F} = \vec{\nabla} \wedge \vec{F} = 0 \quad \rightarrow \begin{cases} \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \\ \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \\ \frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \end{cases}$

5.4 Kinetic Energy:

The kinetic energy of a material point of mass m and speed $\vec{v}$ is :

$$E_c = \frac{1}{2} m v^2$$

Kinetic energy can also be defined in terms of momentum as :

$$E_c = \frac{p^2}{2m}$$

5.5 Potential Energy:

Potential energy is the potential function associated with the conservative force.

Potential energy is defined up to a constant; it is always related to a referential taken as origin to calculate it.
The work of a conservative force $\vec{\boldsymbol{F}}$ is related to the potential energy by the expression :

$$W_{AB} = -\Delta E_p = E_p(A) - E_p(B)$$

Here are some examples of potential energies :
Potential energy of a spring of stiffness K is :

$$E_p = \frac{1}{2}Kx^2$$

Gravitational potential energy of a mass m in the field created by a mass M is :

$$E_p = -G\frac{mM}{r^2}$$

Gravitational potential energy is :

$$E_p = mgz$$

5.6 Work energy theorem:

The variation of the kinetic energy of a material point between two positions A and B is equal to the sum of the works of the forces which are applied to it between these two positions.

$$\Delta E_c = \sum_i W_i$$

5.7 Total Mechanical Energy:

The mechanical energy of a material point is the sum of the kinetic and potential energies :

$$E_T = E_c + E_p$$

5.8 Principle of Conservation of Total Mechanical Energy:

The total mechanical energy of a material point subjected to conservative forces is conserved.

$$(E_T)_{(A)} = (E_T)_{(B)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(B)} + (E_P)_{(B)} \quad \rightarrow \quad \Delta E_T = 0$$

5.9 Non-Conservative Forces:

If one of the forces is not conservative, the mechanical energy is not conserved. In this case :

$$\Delta E_T = \sum_i W_i \, (forces\ non\ conservatives)$$

Exercises :

1. Consider the force $\vec{F} = 8xy\vec{\imath} - (x^2 + y^2)\vec{\jmath}$ applied to a material point moving between two points A(0,1) and B(1,2) :
   - Calculate the work of this force along the path $y = x + 1$
   - Calculate the work of this force along the path $y = x^2 + 1$
   - What can we conclude about strength $\vec{F}$?

- Following the path $y = x + 1 \quad \rightarrow \quad dy = dx$ and $0 \leq x \leq 1$

$$W_{AB}^{(1)} = \int_{(A)}^{(B)} \vec{F} \cdot d\vec{l} = \int_{(A)}^{(B)} (8xydx - (x^2 + y^2)dy = \int_{(A)}^{(B)} (8x(x+1)dx - (x^2 + (x+1)^2)dy$$

$$W_{AB}^{(1)} = \int_0^1 (6x^2 + 6x - 1)dx = 4$$

- Following the path$y = x^2 + 1 \quad \rightarrow \quad dy = 2xdx$ and $0 \leq x \leq 1$

$$W_{AB}^{(2)} = \int_{(A)}^{(B)} \vec{F} \cdot d\vec{l} = \int_{(A)}^{(B)} (8xydx - (x^2 + y^2)dy = \int_{(A)}^{(B)} [8x(x^2 + 1)]dx - [x^2 + (x^2 + 1)^2]2xdx$$

$$W_{AB}^{(2)} = \int_0^1 (-2x^5 + 2x^3 + 6x)dx = \frac{19}{6}$$

- Since the work of the force along the two paths is different, one can conclude that the force $\vec{F}$ is not conservative.

2. Let the force be $\vec{F} = y^2\vec{\imath} + 2xy\vec{\jmath}$. Calculate its work according to the following paths :
   - $O(0,0,0) \rightarrow A(1,0,0) \rightarrow \ B(1,1,0)$
   - The right$O(0,0,0) \rightarrow B(1,1,0)$
   - The connecting parabola $O(0,0,0) \rightarrow B(1,1,0)$: $y = x^2$ in the plane XOY
   - The path closes:$O(0,0,0) \rightarrow A(1,0,0) \rightarrow B(1,1,0) \rightarrow C(0,1,0) \rightarrow O(0,0,0)$
   - Calculate $\overrightarrow{\boldsymbol{rot}}\,\vec{F}$, what can be concluded about the force $\vec{F}$?
   - Calculate the work of $\vec{F}$ of $O(0,0.0) \rightarrow B(1,1,0)$ in general.

- Following the path $O(0,0) \rightarrow A(1,0,0) \rightarrow \ B(1,1,0)$

$$W_{OAB}^{(1)} = W_{OA}^{(1)} + W_{AB}^{(1)} = \int_{(O)}^{(A)} \vec{F} \cdot d\vec{l} + \int_{(A)}^{(B)} \vec{F} \cdot d\vec{l}$$

Next $O(0,0.0) \rightarrow A(1,0,0)$: We have $y = 0 \quad \rightarrow \quad dy = 0\ ,\ dz = 0$ and $0 \leq x \leq 1$

$W_{OA}^{(1)} = \int_{(O)}^{(A)} y^2dx + 2xydy = \int_0^1 0dx - 0dy = 0.$

The force is perpendicular to the displacement OA.

Next $A(1,0,0) \rightarrow \ B(1,1,0)$: We have $x = 1 \quad \rightarrow \quad dx = 0\ ,\ dz = 0$ and $0 \leq y \leq 1$

$$W_{AB}^{(1)} = \int_{(A)}^{(B)} y^2dx + 2xydy = \int_0^1 y^2(0) + 2ydy = 1$$

Therefore, $W_{OAB}^{(1)} = 1$

- Following the line : $O(0,0,0) \rightarrow B(1,1,0)$: $\quad y = x \quad \rightarrow \quad dx = dy \ , \ dz = 0$ and $0 \leq x \leq 1$

$$W_{OB}^{(2)} = \int_{(A)}^{(B)} x^2dx + 2x^2dx = \int_0^1 3x^2dx = 1$$

- Following the parabola : $O(0,0,0) \rightarrow B(1,1,0) : y = x^2 \rightarrow dy = 2xdx \, , dz = 0$ and $0 \leq x \leq 1$

$$W_{OB}^{(3)} = \int_{(A)}^{(B)} y^2dx + 2xydy = \int_0^1 5x^4dx = 1$$

- Following the closed path : $O(0,0,0) \ \rightarrow \ A(1,0,0) \rightarrow \ B(1,1,0) \ \rightarrow \ C(0,1,0) \ \rightarrow \ O(0,0,0)$

$$W_{OABCO}^{(4)} = W_{OA}^{(4)} + W_{AB}^{(4)} + W_{BC}^{(4)} + W_{CO}^{(4)} = \int_{(O)}^{(A)} \vec{\boldsymbol{F}} \cdot d\vec{l} + \int_{(A)}^{(B)} \vec{\boldsymbol{F}} \cdot d\vec{l} + \int_{(B)}^{(C)} \vec{\boldsymbol{F}} \cdot d\vec{l} + \int_{(C)}^{(O)} \vec{\boldsymbol{F}} \cdot d\vec{l}$$

Following the line : $O(0,0,0) \ \rightarrow \ A(1,0,0)$: $\ W_{OA}^{(4)} = W_{OA}^{(1)} = 1$

Following the line : $A(1,0.0) \ \rightarrow \ B(1,1,0)$: $\ W_{AB}^{(4)} = W_{AB}^{(1)} = 1$

Next $B(1,1,0) \ \rightarrow \ C(0,1,0)$: We have $y = 1 \quad \rightarrow \quad dy = 0 \ , \ dz = 0$ and $0 \leq x \leq 1$

$$W_{BC}^{(4)} = \int_{(B)}^{(C)} y^2dx + 2xydy = \int_1^0 dx + 2x(0)dx = -1$$

Next $C(0,1,0) \ \rightarrow \ O(0,0,0)$: We have $x = 0 \quad \rightarrow \quad dx = 0 \ , \ dz = 0$ and $0 \leq y \leq 1$

$W_{BC}^{(4)} = \int_{(B)}^{(C)} y^2dx + 2xydy = \int_1^0 dx + 2x(0)dx = -1$

$W_{OABCO}^{(4)} = W_{OA}^{(4)} + W_{AB}^{(4)} + W_{BC}^{(4)} + W_{CO}^{(4)} = 0$

- $$\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{F}} = \vec{\nabla}\wedge\vec{\boldsymbol{F}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & 0 \end{vmatrix} = (2y - 2y)\vec{k} = 0$$

  We can conclude that the force $\vec{\boldsymbol{F}}$ is conservative. Note that its work is independent of the path. We also notice that the work following a closed path is zero.

- Let's look for the potential function associated with the force $\vec{\boldsymbol{F}}$:

$$\begin{cases} y^2 = -\frac{\partial E_p}{\partial x} \\ 2xy = -\frac{\partial E_p}{\partial y} \\ 0 = -\frac{\partial E_p}{\partial z} \end{cases} \rightarrow \quad \begin{matrix} E_p(x,y,z) = -xy^2 + C(y,z) \\ E_p(x,y,z) = -xy^2 + C(y,z) \\ E_p(x,y,z) = E_p(x,y) \end{matrix}$$

$$E_p(x,y,z) = -xy^2 + C(y) \ \rightarrow \ \frac{\partial E_p}{\partial y} = -2xy + \frac{dC}{dy} \ \rightarrow \ \frac{dC}{dy} = 0 \ \rightarrow \ C(y) = constante\ K$$

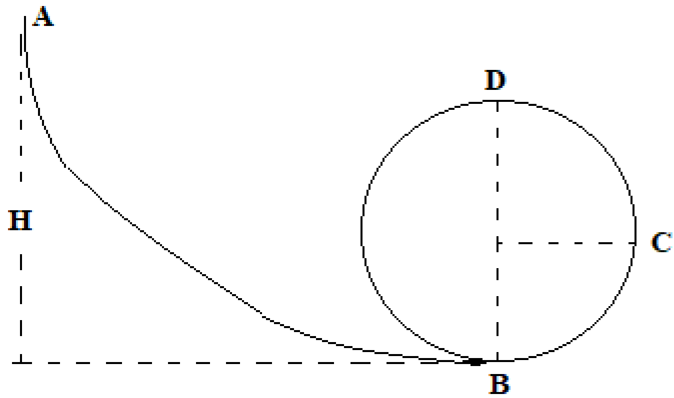
The potential function associated with the force $\vec{\boldsymbol{F}}$ is equal :

$E_p(x,y,z) = -xy^2 + K$

In general, the work of $\vec{\boldsymbol{F}}$ from $O(0,0,0)$ to $B(1,1,0)$ is equal to :

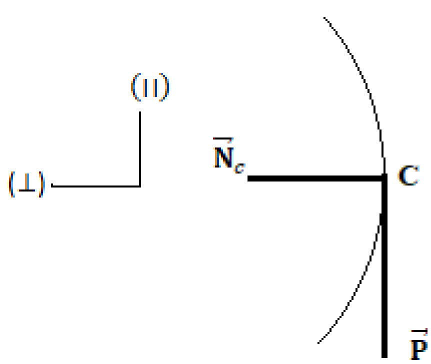
$W_{OB} = -\Delta E_p = E_p(O) - E_p(B) = E_p(0,0,0) - E_p(1,1,0) = 1$

3. A body of mass M is released without initial velocity from the top of a mountain of height H as shown in the figure below :



- Find the velocities of the body M at points B, C and D.
- Find the surface reaction forces on the body M at points C and D.

- In the absence of friction, we have conservation of total mechanical energy :
  $(E_T)_{(A)} = (E_T)_{(B)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(B)} + (E_P)_{(B)}$
  Taking the level (B) as the initial reference of the potential energy $\rightarrow \quad (E_P)_{(B)} = 0$
  Because $v_A = 0 \quad \rightarrow \quad (E_C)_{(A)} = 0$
  $\rightarrow \quad MgH = \frac{1}{2}Mv_B^2 \quad \rightarrow \quad v_B = \sqrt{2gH}$
- $(E_T)_{(A)} = (E_T)_{(C)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(C)} + (E_P)_{(C)}$
  $MgH = \dfrac{1}{2}Mv_c^2 + MgR \quad \rightarrow \quad v_c = \sqrt{2g(H-R)}$
- $(E_T)_{(A)} = (E_T)_{(D)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(D)} + (E_P)_{(D)}$
  $MgH = \dfrac{1}{2}Mv_D^2 + 2MgR \quad \rightarrow \quad v_D = \sqrt{2g(H-2R)}$
- By applying the fundamental principle of dynamics for the body at point C :
  $\vec{\mathbf{P}} + \vec{\mathbf{N}}_c = M\vec{\mathbf{a}}_c$



By projecting on the two axes (II) and (⊥) :

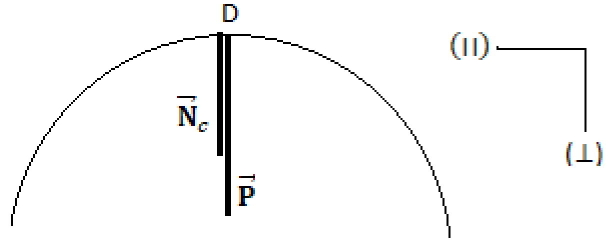
(II) : $Ma_{II} = -Mg$

(⊥) : $Ma_{\perp} = N_c$

$N_c = M\dfrac{v_c^2}{R} = 2\dfrac{(H-R)}{R}Mg$

- By applying the fundamental principle of dynamics for the body at point D :

$$\vec{\mathbf{P}} + \vec{\mathbf{N}}_D = M\vec{\mathbf{a}}_D$$



By projecting on the two axes (II)and (⊥) :

(II) : $Ma_{II} = 0$

(⊥) : $Ma_\perp = Mg + N_D \quad \rightarrow \quad N_D = Ma_\perp - Mg$

$$N_D = M\frac{v_D^2}{R} - Mg = \frac{(2H - 5R)}{R}Mg$$

4. How fast does a rocket need to escape the earth's gravitational field?
   Acceleration of Gravity at Earth's Surface $g_0 = 9.81m/s^2$ and Earth's Radius $R = 6.36x10^6m$
   - By applying the principle of conservation of total mechanical energy :

   $(E_T)_{(I)} = (E_T)_{(F)} \quad \rightarrow \quad (E_C)_{(I)} + (E_P)_{(I)} = (E_C)_{(F)} + (E_P)_{(F)}$

   $$\frac{1}{2}mv_I^2 - G\frac{mM}{R^2} = \frac{1}{2}mv_F^2 - G\frac{mM}{r^2}$$

   The rocket escapes gravity $\rightarrow \quad r = \infty \;\; et \;\; v_F = 0$

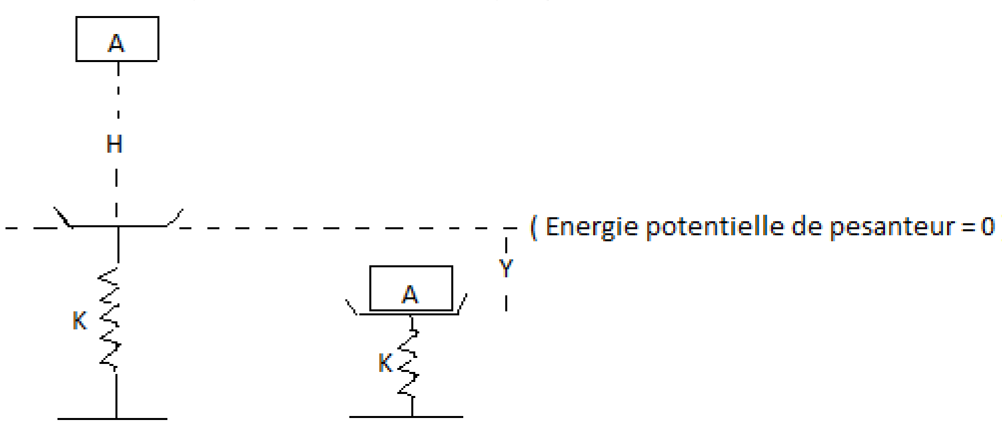
   $$\rightarrow \quad \frac{1}{2}mv_I^2 - G\frac{mM}{R^2} = 0 \quad \rightarrow \quad v_I^2 = 2G\frac{M}{R^2}$$

   The force of weight defined by the universal law of gravitation : $mg_0 = G\frac{mM}{R^2} \quad \rightarrow \quad g_0 = G\frac{M}{R^2}$

   $\rightarrow \quad v_I^2 = 2g_0R$

   N.A : $v_I = 1.12x10^4m/s$

5. A mass M falls from a height H onto a free spring of stiffness K.
   - Find the maximum compression distance of the spring.



- In the absence of friction, we have conservation of total mechanical energy :

$(E_T)_{(I)} = (E_T)_{(F)} \quad \rightarrow \quad (E_C)_{(I)} + (E_P)_{(I)} = (E_C)_{(F)} + (E_P)_{(F)}$

$v_I = v_F = 0 \quad \rightarrow \quad (E_C)_{(I)} = (E_C)_{(F)} = 0 \quad \rightarrow \quad (E_P)_{(I)} = (E_P)_{(F)}$

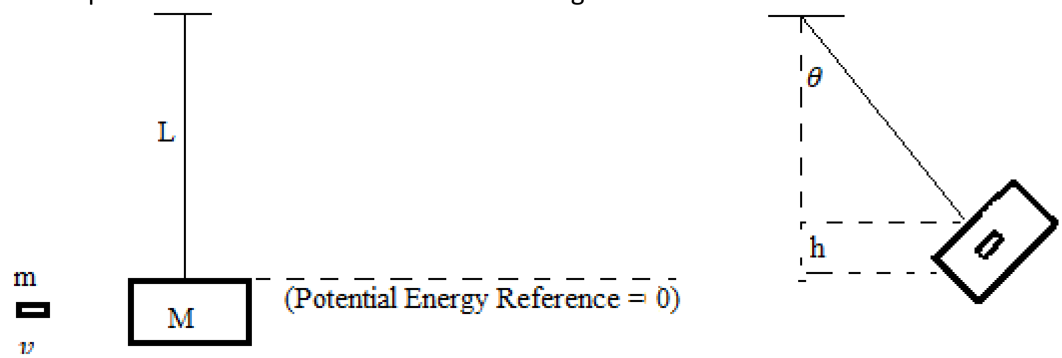
The zero level of the gravitational potential energy is indicated in the figure.

$\rightarrow \quad Mgh = -Mgy + \frac{1}{2}Ky^2 \quad \rightarrow \quad \frac{1}{2}Ky^2 - Mgy - Mgh = 0$

It is a second degree equation. Only the positive root is acceptable.

$$y = \frac{Mg + \sqrt{M^2g^2 + 2MgKh}}{K} = \frac{Mg}{K}(1 + \sqrt{1 + 2\frac{Kh}{Mg}})$$

6. A bullet of mass m is fired from a pistol with velocity v. It hits a block of wood of mass M attached to the end of a wire of length L. The two bodies (Block + ball) move apart at an angle θ.
   - Find the speed of the ball v as a function of the angle θ.



- The conservation of momentum before and after the collision between the ball and the block :

Perfectly inelastic collision : $\rightarrow \quad mv = (m+M)V \quad \rightarrow \quad V = \frac{m}{(m+M)}v$

In the absence of friction, we have conservation of total mechanical energy :

$(E_T)_{(I)} = (E_T)_{(F)} \quad \rightarrow \quad (E_C)_{(I)} + (E_P)_{(I)} = (E_C)_{(F)} + (E_P)_{(F)}$

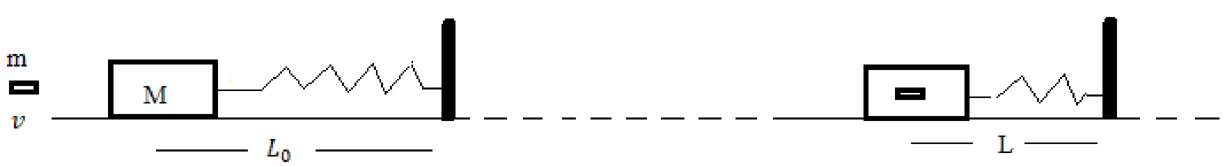
$\frac{1}{2}(m+M)V^2 + 0 = 0 + (m+M)gh \quad \rightarrow \quad h = \frac{1}{2g}V^2 = \frac{1}{2g}[\frac{m}{(m+M)}v]^2$

$\rightarrow \quad v = \frac{(m+M)}{m}\sqrt{2gh}$

$h = L(1 - cos\theta) = 2Lsin^2(\frac{\theta}{2})$

$\rightarrow \quad v = 2\sin(\frac{\theta}{2})\frac{(m+M)}{m}\sqrt{gL}$

7. A bullet of mass m is fired from a pistol with velocity v. It hits a block of wood of mass M attached to a free spring of stiffness K. The dynamic coefficient of friction between the wooden block (M) and the contact surface is μ_d .
   - Find the ball speed as a $v$function of the maximum compression of the spring.



- The conservation of momentum before and after the collision between the ball and the block :

  Perfectly inelastic collision : $\rightarrow \quad mv = (m+M)V \quad \rightarrow \quad V = \frac{m}{(m+M)}v$

  In the presence of friction, there is no conservation of total mechanical energy, we use the work energy theorem :

  $\Delta E_c = (E_C)_{(F)} - (E_C)_{(I)} = \sum_i W_i = W(\vec{\mathbf{P}}) + W(\vec{\mathbf{N}}) + \mathrm{W}(\vec{\mathbf{f}}) + \mathrm{W(ressort)}$

  $(E_C)_{(I)} = \frac{1}{2}(m+M)V^2 = \frac{1}{2}\frac{m^2}{(m+M)}v^2 \quad$ and $\quad (E_C)_{(F)} = 0$

  The sum of the works of the forces $\sum_i W_i = \mathrm{W}(\vec{\mathbf{f}}) + \mathrm{W(ressort)} = -f(L_0 - L) - \frac{1}{2}K(L - L_0)^2$

  $W(\vec{\mathbf{P}}) = W(\vec{\mathbf{N}}) = 0;$ these forces are perpendicular to the displacement.

  Maximum spring compression$(L_0 - L) = X$

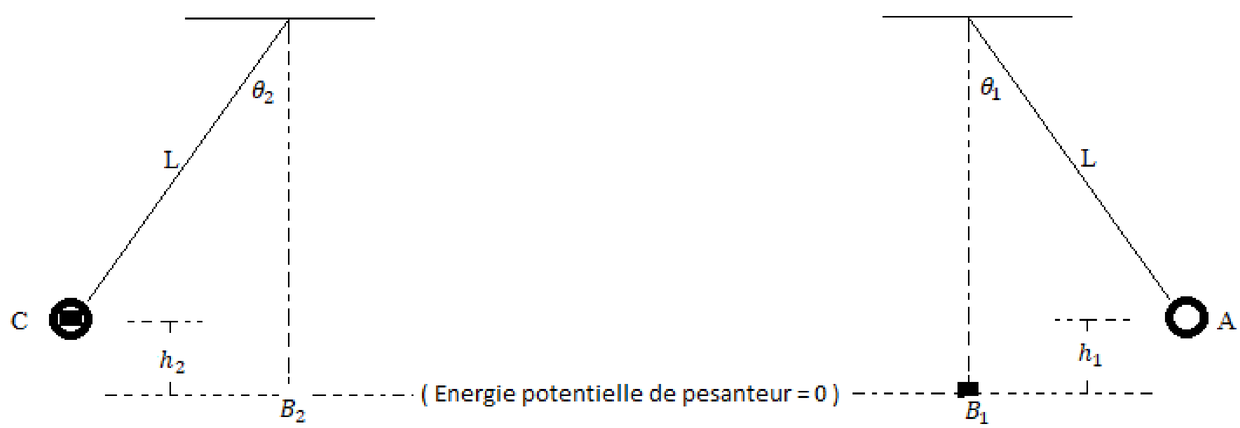
  Force of friction $f = \mu_d(m+M)g$

  $\rightarrow \frac{1}{2}KX^2 + \mu_d(m+M)gX - \frac{1}{2}\frac{m^2}{(m+M)}v^2 = 0$

  The speed of the ball is :

  $\rightarrow v = \frac{(m+M)}{m}\sqrt{2\mu_d gX + \frac{KX^2}{(m+M)}}$

8. A mass M attached to the end of a wire of length L is angled aside $\theta_1$at point A and released without initial velocity. At point B, it strikes and sticks to a small mass m. Both bodies (M + m) deviate with an angle θ_2.
   - Find the relationship between the two angles $\theta_1$and θ_2.



- The fall of the mass M from (A) to (B_1) :

  In the absence of friction, we have conservation of total mechanical energy:

  $(E_T)_{(A)} = (E_T)_{(B_1)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(B_1)} + (E_P)_{(B_1)}$

  $0 + Mgh_1 = \frac{1}{2}MV_{B_1}^2 + 0 \quad \rightarrow \quad V_{B_1} = \sqrt{2gh_1}$

- The collision between M and m :

  The conservation of momentum before and after the collision between the ball and the block :

Perfectly inelastic collision : $\rightarrow \; V_{B_1} = (m+M)V_{B_2} \quad \rightarrow \; V_{B_2} = \frac{m}{(m+M)}V_{B_1} = \frac{m}{(m+M)}\sqrt{2gh_1}$

- Movement of the two masses M and m from (B_2) to (C) :

  $(E_T)_{(B_2)} \; = \; (E_T)_{(c)} \quad \rightarrow \quad (E_C)_{(B_2)} + (E_P)_{(B_2)} = (E_C)_{(c)} + (E_P)_{(c)}$

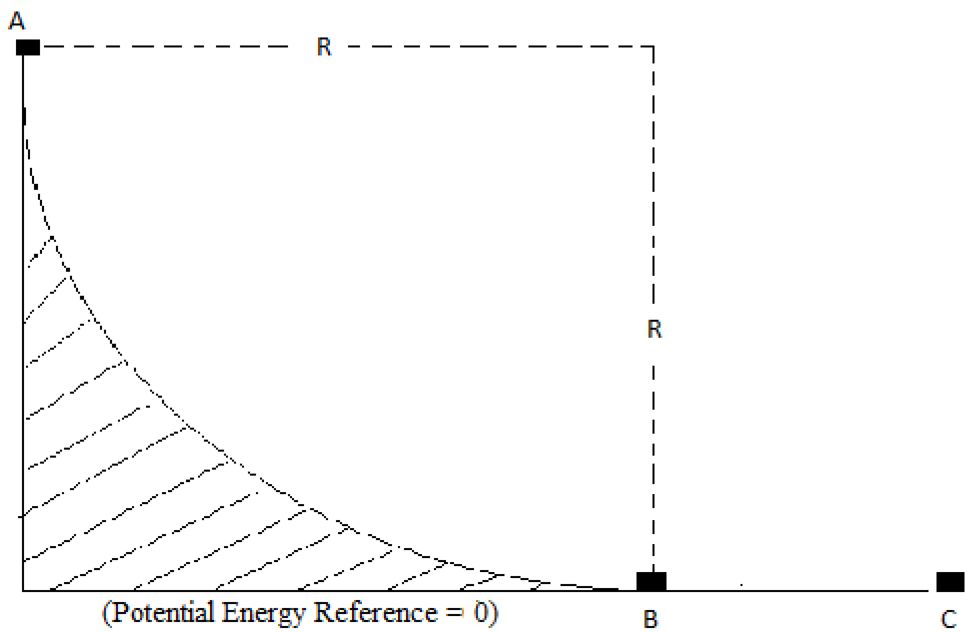
  $\frac{1}{2}(m+M)V_{B_2}^2 + 0 = 0 + (m+M)gh_2 \quad \rightarrow \; h_2 = \frac{V_{B_2}^2}{2g} = (\frac{M}{m+M})^2 h_1$

  $h_1 = L(1-cos\theta_1) = 2Lsin^2\left(\frac{\theta_1}{2}\right)$ and $h_2 = L(1-cos\theta_2) = 2Lsin^2\left(\frac{\theta_2}{2}\right)$

  The relationship between the two angles $\theta_1$and θ_2 is :

  $sin\left(\frac{\theta_2}{2}\right) = (\frac{M}{m+M})sin\left(\frac{\theta_1}{2}\right)$

9. A body of mass M is released without initial velocity from the top of a quadrant of radius R as shown in the figure below. The body is subjected to a frictional force of constant magnitude f throughout its journey. It reaches point B with speed v and stops at point C.



- Find the magnitude of the frictional force $\vec{\mathbf{f}}$.
- Find the distance BC.

- In the presence of friction, there is no conservation of total mechanical energy, we use the work energy theorem on the quarter circle :

  $\Delta E_c = (E_C)_{(B)} - (E_C)_{(A)} = \sum_i W_i = W(\vec{\mathbf{P}}) + W(\vec{\mathbf{N}}) + \mathrm{W}(\vec{\mathbf{f}})$

  The reaction force $\vec{\mathbf{N}}$ is perpendicular to the displacement : $W(\vec{\mathbf{N}}) = 0$

  $\rightarrow \quad \frac{1}{2}Mv^2 - 0 = MgR - f\frac{\pi R}{2}$

  $\rightarrow \quad f = \frac{2}{\pi R}[MgR - \frac{1}{2}Mv^2]$

- On the course (BC) : $\Delta E_c = (E_C)_{(C)} - (E_C)_{(B)} = \sum_i W_i = W(\vec{\mathbf{P}}) + W(\vec{\mathbf{N}}) + \mathrm{W}(\vec{\mathbf{f}})$

  The forces $\vec{\mathbf{P}}$ and $\vec{\mathbf{N}}$ are perpendicular to the displacement : $W(\vec{\mathbf{P}}) = W(\vec{\mathbf{N}}) = 0$

$$\rightarrow \quad 0-\frac{1}{2}Mv^2=-fD \quad \rightarrow \quad D=\frac{Mv^2}{2f}$$

The stopping distance BC is equal to : $D=\frac{\pi R v^2}{4gR-2v^2}$

10. A water pump has a power $P = 12\ hp$. It is used to pump water from a well of depth H with a flow rate of $\frac{\Delta V}{\Delta t} = 2\frac{m^3}{mn}$.

   - What is the maximum depth of the point if the efficiency of the pump is $\varepsilon = 80\%$.
   - The pump does work per unit time equal to the potential energy of the water's gravity per unit time. The power required to pump water from a depth H is therefore equal to :

$$\varepsilon P=\frac{\Delta W}{\Delta t}=\frac{(\Delta m)gH}{\Delta t}=\frac{\Delta m}{\Delta t}gH=\frac{\rho\Delta V}{\Delta t}gH$$

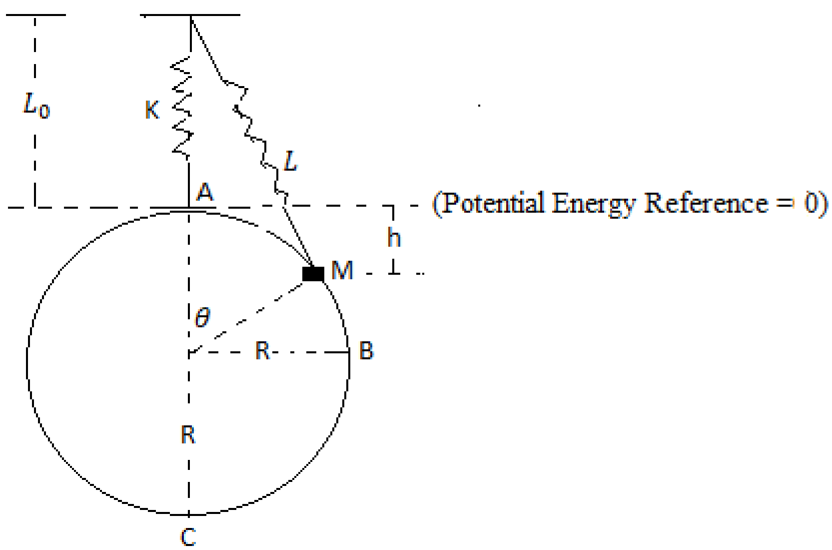
$$\rightarrow \quad H=\frac{\varepsilon P}{\rho g\frac{\Delta V}{\Delta t}}$$

N.A : $P = 12\ hp = 12x746 = 8952\ Watts$

$H \approx 22\ m$

11. A body of mass M is attached to a spring of stiffness K as shown in the figure below. The spring is free and not stretched at point A. If the velocity of the body at point C is v_c.

   - Find the velocity of the body at point B and A.



- In the absence of friction, we have conservation of total mechanical energy :

$$(E_T)_{(M)}=(E_T)_{(c)} \quad \rightarrow \quad (E_C)_{(M)}+(E_P)_{(M)}=(E_C)_{(c)}+(E_P)_{(c)}$$

$$(E_T)_{(c)}=(E_C)_{(c)}+(E_P)_{(c)}=\frac{1}{2}Mv_c^2-2MgR+\frac{1}{2}K(2R)^2=\frac{1}{2}Mv_c^2-2MgR+2KR^2$$

$$(E_T)_{(M)}=(E_C)_{(M)}+(E_P)_{(M)}=\frac{1}{2}Mv^2-Mgh+\frac{1}{2}K(L-L_0)^2$$

$$\frac{1}{2}Mv^2-Mgh+\frac{1}{2}K(L-L_0)^2=\frac{1}{2}Mv_c^2-2MgR+2KR^2$$

$$\rightarrow \quad v = \sqrt{v_c^2 + \frac{K}{M}[4R^2 - (L - L_0)^2] - 2g(2R - h)}$$

$h = R(1 - cos\theta) = 2Rsin^2(\frac{\theta}{2})$

$L = \sqrt{(L_0 + R)^2 + R^2 - 2R(L_0 + R)cos\theta}$

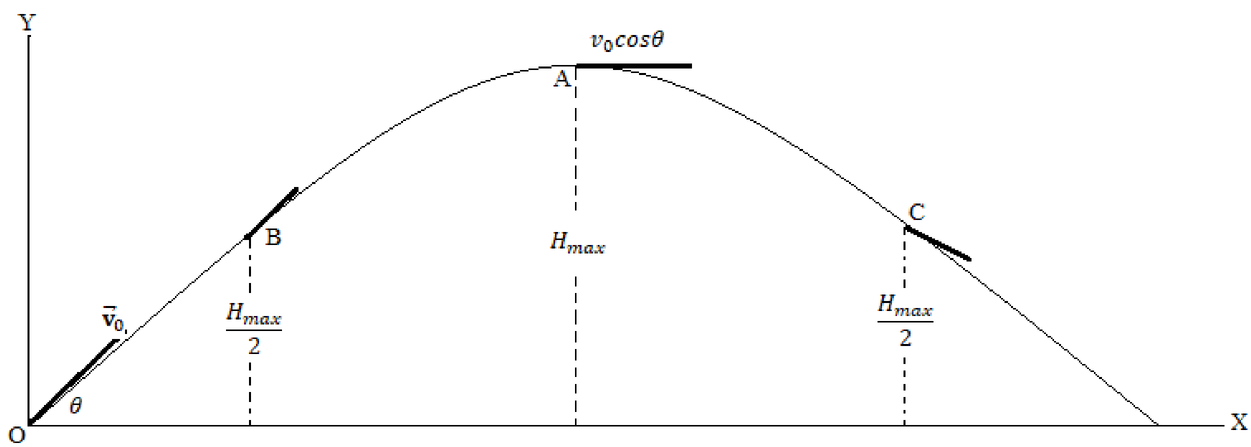
- at point B : $\theta = \frac{\pi}{2}$, $h = R$ and $L = \sqrt{(L_0 + R)^2 + R^2}$

$$v_B = \sqrt{v_c^2 + 2\frac{K}{M}\left[R^2 - L_0^2 - RL_0 + L_0\sqrt{L_0^2 + 2RL_0 + 2R^2}\right] - 2gR}$$

- at point A : $\theta = 0$, $h = 0$ and $L = L_0$

$$v_A = \sqrt{v_c^2 + \frac{4K}{M}R^2 - 4gR}$$

12. A projectile is launched with a magnitude velocity v_0 directed at an angle θ to the horizontal.
    - Calculate the maximum height H_{max} reached by the projectile.
    - Calculate the speed of the projectile when it has reached half of its greatest height.



- In the absence of friction, we have conservation of total mechanical energy :

  $(E_T)_{(O)} = (E_T)_{(M)} \quad \rightarrow \quad (E_C)_{(O)} + (E_P)_{(O)} = (E_C)_{(M)} + (E_P)_{(M)}$

  The maximum height H_{max} is reached by the projectile at point A when : $v_A = v_0cos\theta$

  $\frac{1}{2}Mv_0^2 + 0 = \frac{1}{2}Mv_0^2cos^2(\theta) + MgH_{max}$

  $$\rightarrow \quad H_{max} = \frac{v_0^2(1 - cos^2(\theta))}{2g} = \frac{v_0^2sin^2(\theta)}{2g}$$

- $(E_T)_{(O)} = (E_T)_{(B)} \quad \rightarrow \quad (E_C)_{(O)} + (E_P)_{(O)} = (E_C)_{(B)} + (E_P)_{(B)}$

  $\frac{1}{2}Mv_0^2 = \frac{1}{2}Mv_B^2 + Mg\frac{H_{max}}{2} \quad \rightarrow \quad v_B = \sqrt{v_0^2 - gH_{max}}$

  $$\rightarrow \quad v_B = v_0\sqrt{1 - \frac{sin^2(\theta)}{2}}$$

  $v_B = v_c$ There are two positions B and C with the same velocity at height : $H = \frac{H_{max}}{2}$

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